Tournaments, Kings, and Realizable Posets

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Abstract

The intent of this paper is summarize results from an exploration of which posets, defined according to the relation of unkingliness, are realizable as tournaments. This exploration was done in collaboration with Abel Romer. In particular, the algorithm in section 4 was originally written by Romer in python. Most of our exploration thus far has been based on the observation that all posets can be represented as a matrix (as shown by [3]). We prove some properties, show examples of realizable and non-realizable posets, characterize a class of realizable posets, and propose next steps for the exploration.

1 Introduction

Below are preliminary terms that are used throughout the document.

1.1 Basic Terms

Definition 1.1 (*Tournament*)**.** Let T be a directed complete graph on n vertices contained in set $V(T)$ with arcs in $E(T)$. T is called a tournament.

Definition 1.2 (Kings and Emperors)**.** Let T be a tournament. If some $v \in V(T)$ has $d^-(v) = n-1$, then v reaches every other vertex in 1 step and v is called an **emperor** of T. If v reaches every other vertex of T within at most 2 steps, then v is called a **king** of T.

Definition 1.3 (Score Sequence)**.** Let T be a tournament. Then the score sequence of T is the non-increasing integer sequence corresponding to the out-degrees of each vertex beginning with a vertex of highest degree.

Definition 1.4 (Poset)**.** Let $V(T)$ be the ground set with relation $\leq$ defined as follows: $A \leq B$ iff A cannot reach B in at most 2 steps on T. Then $P = (V(T), \leq)$ is the poset we are concerned with in this document. Unless otherwise stated, all posets will be of this form relative to $\leq$. We call $\leq$ the **unkingliness** relation.

Definition 1.5 ([3] Poset Matrix)**.** Any poset $(P, \leq)$ has an associated upper triangular matrix M with 1s on the main diagonal where

1. $(m_i, m_j) = 1$ in the matrix iff $i \leq j$, where $i, j \in P$.
2. (transitivity:) if there exists i, j, and $k \in P$ such that $i \leq j$, and $j \leq k$, then it must be true that $(i, k) = 1$ since $i \leq k$.

Definition 1.6. Let $(P, \leq)$ be a poset. A poset path is a sequence of P elements such that for each consecutive pair in the sequence $\{A, B\}$, $A \leq B$.

2 Relationships Between Tournaments, Kings, and Posets

J.W. Moon in [1] proved that every vertex that doesn't dominate every other vertex in a tournament is dominated by a king. This implies that there exists a king in every tournament.

Proposition 2.1. *A vertex v is a king in T if and only if the row in a poset matrix corresponding to v contains all 0s except for a single 1 on the main diagonal.*

Proof. Let v be a king in T, and M be the poset matrix corresponding to $(V(T), \leq)$. Non-main-diagonal entries correspond to non-reflexive pairs in $\leq$ of the form $i \leq j$ where $i \neq j$, $i, j \in V(T)$. Suppose a non-main-diagonal entry in the row of M corresponding to v, call it r, and some column s has value 1, that is, $m_{r,s} = 1$, then it follows that $r \leq s \in (P, \leq)$, implying that r cannot reach s in at most 2 steps on T, which contradicts the assumption that v is a king.

Conversely, suppose we have a poset matrix realized by T and the row r corresponding to v has all 0 entries except for a 1 on the main diagonal. Then by the the unkingliness relation, it follows that there does not exist another vertex u in T besides v such that v cannot reach it in at most 2 steps on T. This implies that v is a king in T. □

Proposition 2.2. *These facts follow immediately from the definition of $\leq$:*

1. *If $A \leq B$ for some $A, B \in V(T)$, then A is not a king of T.*
2. *$A \leq B$ implies that $BA \in E(T)$.*
3. *If $A \leq B$ and $B \leq C$, then BA, CB, and CA are all in $E(T)$.*

Proof.

1. If $A \leq B$, then A cannot reach other vertex in T, thus A is not a king.
2. Suppose $A \leq B$. Exactly one of AB or $BA \in E(T)$, if $AB \in E(T)$, then $A \nleq B$, a contradiction, therefore $BA \in E(T)$.
3. By transitivity, the assumption implies $A \leq C$, and by the immediately previous fact, it follows that BA, CB, and CA are in $E(T)$.

□

3 Poset and Tournament Matrices

Proposition 3.1. *Let A, B be elements in an arbitrary poset P with poset matrix M where rows and columns i and j correspond to the relations involving A and B, respectively. Then the number of poset paths from A to B is given by the entry of M^2 in the i-th row and j-th column.*

Proof. Let M be an n by n poset matrix for poset $(P, \leq)$. $m_{i,j} = 1$ iff $A \leq B$ and 0 otherwise. If an entry $m_{r,j}$ in the j-th column is 1, then there is a poset path of length 1 beginning at the element of P corresponding to the row r and ending at B. Similarly if an entry $m_{i,c}$ in the i-th row is 1, then there is a poset path of length 1 beginning at A and ending at the element corresponding to the column c. The value of the entry $m'_{i,j}$ of M^2 is the dot product of the i-th row and j-th column of M. Since M is a binary matrix, the terms in the dot product of any row and column will always be 1s. The only time a term is included in the dot product of the i-th row and j-th column of M is when there is a path that begins at A, represented by $m_{i,t} = 1$ and ends at B, represented by $m_{t,j}$. Thus the number of terms in the dot product calculated at $m'_{i,j}$, is the total number of paths both beginning at A and ending at B. Since A and B is an arbitrary pair of elements of P, we are done. □

Example:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

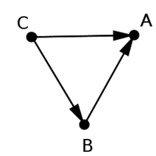


Figure 1: Example showing the relationship between poset matrix M, its square showing the path lengths between vertices A, B, and C in the tournament at the bottom.

4 Posets Not Realizable as Tournaments

Non-realizable posets are difficult to characterize. However they can be determined using the following algorithm (thanks to Romer):

1. Enumerate the adjacency matrices of all possible tournaments on n vertices.
2. Square each of these matrices to determine the path lengths between each pair of vertices.
3. For each squared adjacency matrix A^2, create a new matrix P identified with the identity matrix. If the (i, j) entry in A^2 is 0, then set (i, j) in P to 1.
4. The entries in P then satisfy the rules of the unkingliness relation, so it corresponds to a poset realizable by a tournament.
5. Let U be the set of all upper triangular (0,1)-matrices with 1s along the main diagonal.
6. Remove any matrices in U that do not satisfy the transitivity property: if $u_ij = 1$ and $u_jk = 1$, then $u_ik = 1$. U thus contains all poset matrices.
7. Any matrix now in U that is not equal to a poset matrix P as constructed above therefore does not represent a poset realizable as a tournament.

Using the above algorithm, the following matrices are the poset matrices representing posets unrealizable by tournaments on 3 vertices:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

5 Posets Realizable as Tournaments

Using steps 1 through 4 of the above algorithm, the poset matrix representations of the two realizable posets with 3 elements were found:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

similarly for $n = 4$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & 1 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$$

Proposition 5.1. *If a poset matrix M has 1s above and to the right of the main diagonal, then the corresponding poset is realizable as a tournament according to* $\leq$

Proof. The poset corresponding to M has all possible non-reflexive pairs such that there are no cyclic poset paths. In particular, for every $i \in [1, n]$ there is an element of the ground set that is $\leq$ exactly $n - i$ other elements in the ground set. This poset matrix is clearly transitive since all entries above the main diagonal are 1. In terms of the tournament, for each $i \in [1, n]$ there is a vertex in $V(T)$ that can reach all but $i - 1$ other vertices in at most two steps. We can construct such a tournament as follows:

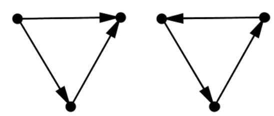
1. Begin with K_n.
2. Choose any vertex in K_n v and create an arc beginning at v ending at every other of the $n - 1$ vertices.
3. For each remaining vertex w, with out degree of 0, create an arc from w to every other possible vertex u so long as uw is not already an arc.

4. Since each step reduces the number of outgoing arcs creatable at proceeding vertices by 1, this orientation is possible to impose, and the resulting tournament will have score sequence of the form $(n-1, n-2, ..., 1)$

□

6 Observations that may be significant

1. Let T be a tournament on n vertices, then every 3-subset of vertices of T induces one of exactly two tournaments on 3 vertices as shown below, one of which, T_1, has an emperor, and the other, T_2 does not:



We could characterize a class of realizable posets by adding rows and columns to the poset matrices corresponding to T_1 and T_2

2. A partial ordering could be defined on the set of realizable posets as follows: for realizable posets A and B, $A \leq B$ iff the tournament that realizes A is an induced subtournament of the tournament that realizes B. Studying this poset may be fruitful since it could suggest a method of constructing realizable posets from other realizable posets. This would help to characterize non-realizable posets.

Acknowledgement

I want to thank Abel for our productive and fun collaboration, and Dr. Huang for suggesting this interesting problem!

7 References

[1] J.W. Moon (1962), Solution to problem 463. Math. Mag., 35, 189

[2] Y. Yu, "Kings in tournaments(2)", Mathematical Medley, 33(2), 15-18

[3] S.U. Mohammad, R. Talukder (2020). Poset Matrix and Recognition of Series-Parallel Posets. International Journal of Mathematics and Computer Science, 15,no. 1, pp. 107-125