List Colourings Project

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1 Introduction

This document introduces notions relating to vertex list colourings. We begin with a discussion on Rubin's [1] characterization of 2-choosable graphs followed by an exploration of list colourings on trees, bipartite, and split graphs. We will end on the 5-choosability of planar graphs theorem with Thomassen's elegant proof and preliminary comments.

1.1 Basic Definitions

Graphs and Proper Colouring Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. A c-colouring of G is a function $f : V(G) \to \{1, ..., k\}$. f is called **proper** if and only if $\forall v \in V(G)$, and $u \in N(v), f(v) \neq f(u)$, otherwise f is called **improper**.

Definition (k-list-colouring) A k list-colouring of G involves a pair of functions: a list function L and colouring function f where L assigns to each vertex a set of colours with size at most k, and f maps each vertex to one of the colours in its list. G is said to have a k-list-colouring if and only if f forms a proper colouring on G.

Definition (k-choosable) A graph G is k-choosable if and only if G has a list-colouring where $\forall v \in V(G), |L(v)| \leq k$ for every list assignment L.

If there exists a list assignment L that assigns a set of colours of size at most k such that the vertices of G are not properly colourable by the colours in their lists, then G is not k-choosable.

Definition (List Chromatic Number) The least number k such that a graph G is k-choosable is called the list-chromatic number, denoted, $\chi_l(G)$. In general, $\chi(G) \leq \chi_l(G)$.

2 2-choosable Graphs

[1] characterizes 2-choosable graphs as those with core being either a K_1, even cycle, or a $\theta_{2,2,2m}$ graph. They prove this after stating that a graph is 2-choosable if and only if its core is 2-choosable, which they seem to assume obvious and so don't show.

Definition (Core of a Graph) The core of a graph G is an induced subgraph obtained by iteratively removing all vertices of degree 1 from G until no such vertex exists. The resulting graph is called the **core** of G.

Proposition A graph G is 2-choosable if and only if its core is 2-choosable.

Proof: Suppose G is 2-choosable, then since its core is simply an induced subgraph of G, it follows that its core is also 2-choosable. Conversely, suppose G's core X is 2-choosable, then since X is the result of iteratively pruning degree 1 vertices from G, we can perform the reverse process by adding appropriate vertices to X until obtaining G. Whenever such a vertex is added, we can assign any list of size at most 2 to it, and no assignment will inhibit the proper colouring of the resulting graph. $\square$

Definition (Theta Graph) A **theta graph**, $\theta_{i,j,k}$ is a graph consisting of exactly three vertex disjoint paths P_i, P_j, and P_k, all beginning at the same vertex u and ending at the same vertex v.

The theta graph used in the 2-choosable characterization by A.L. Rubin takes the following form: $\theta_{2,2,2m}$.

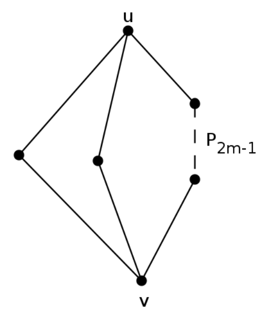


Figure 1: When $m = 1$, all vertex disjoint paths connecting u and v are P_2s because $P_{2(1)-1} = P_1$.

These $\theta_{2,2,2m}$ graphs are 2-choosable. In the worst case, all vertices have the same list of size two $\{x, y\}$, and the middle vertices in the two paths of size 2 must have the same colour, implying that u and v vertices have to be coloured the same. $\theta_{2,2,2m}$ is not 2-choosable if P_{2m-1} is not 2-colourable with beginning and end points having the same colour, but it is since $2m - 1$ is odd; and when $m = 1$, its endpoints are the same vertex, which can be coloured with the same colour as the middle vertices in the other P_2 paths. The proper colouring only fails when u and v are coloured differently and every vertex is assigned the same list of size 2, but if this happens, the colours of u, v, and the two middle vertices of the P_2s can be reconfigured so that u and v are coloured the same.

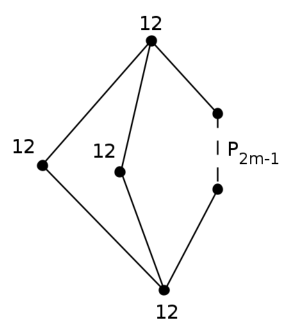


Figure 2: In the worst case, every vertex has list $\{1, 2\}$.

$\mathbf{K_1}$ **and Even Cycles are 2-choosable** Obviously K_1 is 2-choosable. But what about even cycles? Even cycles are 2-choosable because suppose otherwise, then there exists a list assignment on the vertices such that one vertex v is left without a choosable colour. The only way for v to be uncoloured is if its neighbouring vertices x and y are both coloured with the distinct colours in $L(v) = \{r, s\}$. What's more, it is impossible for alternative colours for x and y to be chosen, implying that $L(x) = L(y) = L(v)$, and by transitivity, all vertices in the cycle must have list $\{r, s\}$ otherwise an alternative colour could be chosen for some vertex, enabling a reconfiguration of the colouring to allow v to be properly coloured. But since even cycles are 2-colourable, and each vertex has the same colours to choose from, it follows that the even cycle can be coloured with these lists, which is a contradiction. So, Even cycles are 2-choosable.

3 List Colourings on Various Graph Classes

3.1 Trees are 2-choosable

Since any graph is 2-choosable if its core is K_1, and iteratively removing leaves of a tree until no leaves exist always results in a single vertex, a K_1, it follows that trees are 2-choosable.

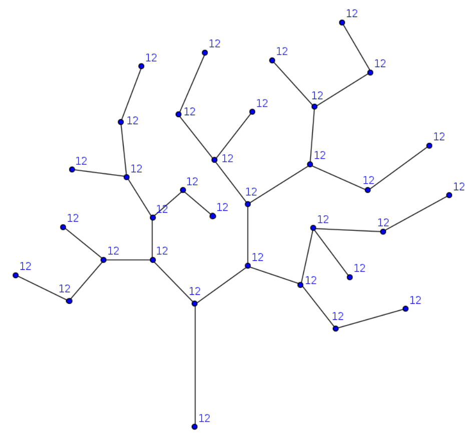


Figure 3: This tree is here because it is nice to look at Fibonacci trees! Also, it is 2-choosable.

3.2 Bipartite Graphs

In general, $\chi(G) \leq \chi_l(G)$, and for some graphs, $\chi_l(G)$ can be arbitrarily larger than $\chi(G)$, as in the case for the following class of complete bipartite graphs:

Complete Bipartite Graph Theorem [1] Let $G = K_{m,m}$ with $m = \binom{2k-1}{k}$, then G is not k-choosable.

Proof [3] It is sufficient to show that there exists a list assignment with each list of size at most k such that a proper colouring of $K_{m,m}$ is not possible.

1. Let X and Y be the bipartition of G.
2. For both X and Y, assign to each vertex a distinct k-subset of $[2k-1]$ as the vertices lists.
3. If less than k colours are chosen to colour the vertices in X, then there exists a list of size k assigned to a vertex in X that does not contain any of these colours chosen to colour the vertices in X. So, this vertex is not properly coloured.
4. If at least k colours are chosen to colour the vertices in X, then these k colours form a k-subset that is equivalent to a list assigned to some vertex y in Y. Since G is complete, the vertices in the neighbourhood of y are coloured with all the colours in y's list, therefore y cannot be coloured properly.
5. Thus G is not k-choosable. □

3.3 Split Graphs

A split graph is a graph whose vertices can be partitioned into two sets X and Y where one set forms a clique and the other forms an independent set.

Recall that the chromatic number of a complete graph on n vertices is n. For any graph G, it is always true that $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of G's maximum clique. If G is a split graph, then G's vertices can be partitioned into a set X containing a clique and an independent set Y. Either $|X| = \omega(G), |Y| = n - \omega(G)$ when X contains a maximum clique, or $|X| = \omega(G) - 1, |Y| = n - \omega(G) + 1$ when exactly one of the vertices of a maximum clique of G is in Y. If we have the latter case, then we can move the vertex in Y that is part of a maximum clique in G into X to ensure X contains a maximum clique of G. Without loss of generality, we will assume that the vertices are organized in such a way to ensure X contains a maximum clique of G.

Claim If G is a split graph then $\chi(G) = \omega(G) = \chi_l(G)$.

Explanation Suppose $\chi(G) > \omega(G)$ then we need more colours due to the vertices in the independent set Y rather than X since $\chi(X) = \omega(G)$. If any one of the vertices y in Y are adjacent to all vertices in X, then y is part of a maximum clique of G, which contradicts our definition of X, so no $y \in Y$ is adjacent to all vertices in X. Thus all vertices in Y are $\omega(G)$-colourable, implying that G is $\omega(G)$-colourable. Obviously, $\chi(G) \nless \omega(G)$, so $\chi(G) = \omega(G)$ when G is a split graph.

A similar argument shows that $\chi_l(G) = \omega(G)$, the vertices in Y will always have enough colour choices to be coloured properly. But while $\chi_l(G) = \omega(G)$, can the list sizes for the vertices in X or Y be smaller than $\omega(G)$? The answer is obviously no for the vertices in X, but less obvious is the answer for the pairwise independent vertices in Y.

Question Can it be true for some split graphs G that $\chi_l(Y) < \omega(G)$?

Counter examples: When all vertices in Y have degree $\omega(G) - 1$, $\chi_l(Y) \nless \omega(G)$ in general. Here are counter examples:

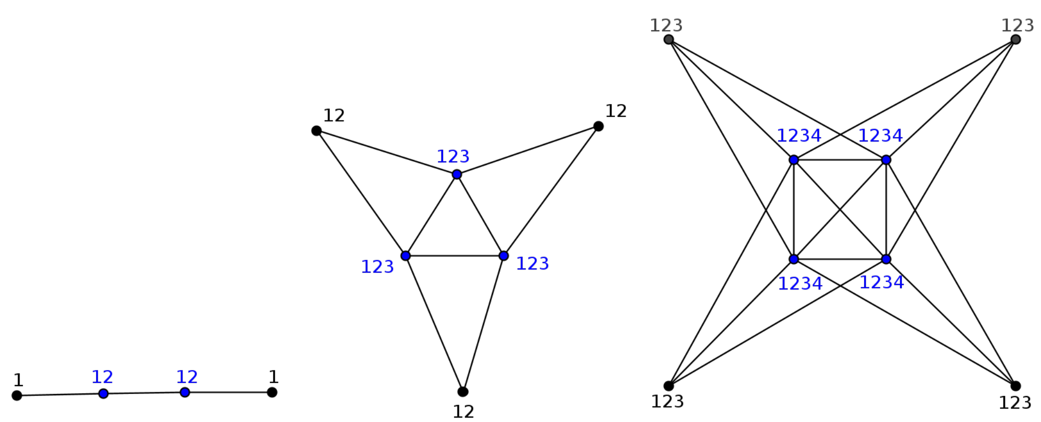


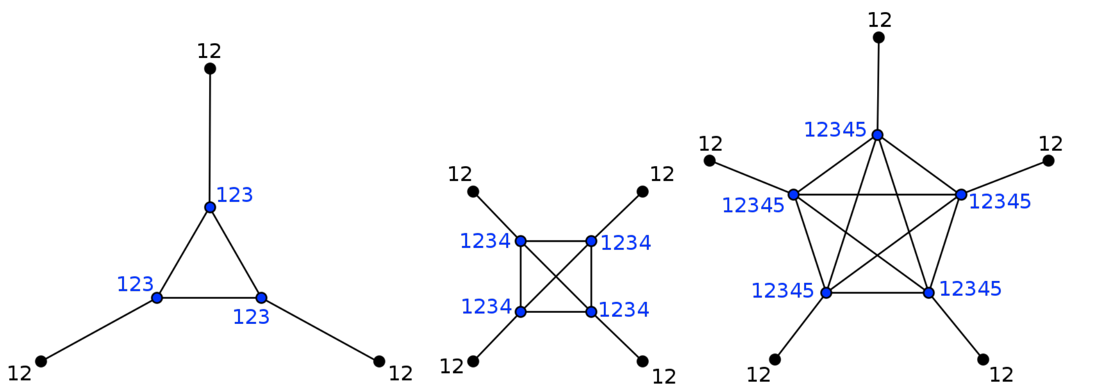
Figure 4: The outer vertices in these graphs form the independent sets Y and their inner vertices form the cliques X. These graph are not properly colourable given the colour lists assigned to their vertices.

Construction of the Above Counter Examples The above family F_n of graphs are created as follows:

1. Let $X = K_n$
2. For each $(n-1)$-subset of vertices in X make all these vertices adjacent to a new vertex y, and include y into a set Y.
3. The result is a split graph with clique X and independent set Y, call it F_n.

The Y vertices in F_n must have colour list size at least n in order to be properly colourable given any list assignment. Otherwise it will always be the case that an $(n-1)$-subset of the X vertices will be coloured by the colours in the list of the Y vertex incident to vertices in this $(n-1)$-subset.

When it does work But for some split graphs, $\chi_l(Y) < \omega(G)$. While the existence of the examples below don't demonstrate this for any list assignment, it isn't hard to see that any list assignment with $|L(y)| = 2, \forall y \in Y$, works for the following types of split graphs:



3.4 Planar Graphs

Definition (Planar Graph) A graph G is a **planar graph** if and only if G can be embedded on the plane, that is there exists a drawing of G on the plane such that no edge intersects another edge except at a vertex. Such a drawing of a planar graph is called a **plane graph** of G.

Definition (Maximally Planar Graph) A planar graph G is **maximally planar** if and only if every face of the graph is bounded by exactly 3 edges. Maximally planar graphs have the most number of edges possible for their number of vertices such that the graph is still planar - if at least one more edge is added to a maximally planar graph, then the resulting graph is not planar.

List Colourings and Planar Graphs In 1994, Carsten Thomassen proved that planar graphs are 5-choosable. Thomassen proves the theorem for the worst case of maximally planar graphs since planar graphs with less edges could have smaller list-chromatic number. The proof uses the following structural features of maximally planar graphs:

- The entire graph is bounded by an outer cycle of vertices, called external vertices. The rest of the vertices are internal vertices
- The vertices in this outer cycle are 3-choosable
- Exactly two of the external vertices can have an arbitrary fixed colour

Thomassen uses these features above in the inductive hypothesis, thereby proving a stronger claim. But the value in formulating this detailed inductive hypothesis does more than merely prove a slightly stronger claim, it allows for an elegant inductive step showing non-obvious properties of the structure to be essential.

3.4.1 Theorem

Planar graphs are 5-choosable.

Proof: [**2**] It is sufficient to prove the case for planar graphs whose internal faces are bounded by exactly three edges - such a graph is called a maximally planar graph. The proof proceeds by induction on the vertices of a maximally planar graph G.

Inductive Hypothesis: G is a 5-choosable maximally planar graph with outer cycle C. G has two external vertices with arbitrary but distinct lists of size 1, and the other external vertices have arbitrary lists of size 3.

Base Case: When $n = 3$, since all vertices are external, we can arbitrarily set two to have distinct lists of size 1 and the other vertex can be assigned any list of size 3. Regardless of the list assignment, these vertices can be properly coloured.

Inductive Step: Let v_1 and v_p in C be the two vertices with fixed colours, and let $v_1, ..., v_p$ be the vertices of C in cyclic ordering. There are two cases. First is when C has a chord, and second, when C is chordless.

C1: Suppose C has a chord v_iv_j with $1 \leq i \leq j-2 \leq p-2$. Then we get a plane graph G' with external cycle $v_1, ..., v_i, v_j, ..., v_p$ and its interior. Clearly G' has less vertices than G, so we can apply the inductive hypothesis to G', which gives us a proper colouring on G' with distinct colours chosen for v_1 and v_p. The presence of the chord also gives us another plane graph

G^* with external cycle $v_i, v_{i+1}, ..., v_j$. We can similarly apply the inductive hypothesis to G^* with v_i and v_j being the two external vertices with fixed colours, since they were coloured after applying the inductive hypothesis to G'. Therefore vertices of G^* and G' are properly coloured with their lists being of size at most 5, so G is 5-choosable.

C2: Suppose C is chordless. v_2 is adjacent to the external vertices v_1 and v_3, and let $u_1, ..., u_m$ be the internal vertices adjacent to v_2 (these are internal because G is chordless). Since G is maximally planar, G contains a path P with vertices $v_1, u_1, ..., u_m, v_3$. Define $G' = G - v_2$, then the external cycle of G' is $P, ..., v_p, v_1$. Let c be the colour assigned to v_1. Since $|L(v_2)| \geq 3$, we can choose one of two distinct colours $x, y \in L(v_2)$ distinct from c. We can then forbid ourselves from choosing x and y in $L(u_i)$ for any $i \in [1, m]$, and since $|L(u_i)| \geq 5$, we can assume that $|L(u_i) - \{x, y\}| \geq 3$. Since these u_i vertices are in the external cycle of G', we can apply the inductive hypothesis to G' with v_1 and v_p having fixed colours, the u_is being 3-choosable, and the other vertices have list sizes the same as in G. The inductive hypothesis gives us a proper colouring of all vertices in G', in particular: $v_1, u_1, ..., u_m$ with at most three colours other than $\{x, y\}$. Then we can add the vertex v_2 to G' and replace its edges as in G. Since none of the neighbours of v_2 have been coloured with either x or y, (recall $c \notin \{x, y\}$ by definition), we can colour v_2 with the colour x or y that is not chosen for v_3 and $|L(v_2)| = 3$. So G is properly coloured with each vertex having colour list size at most 5.

So, all planar graphs are 5-choosable. □

References

[1] P. Erdos, A.L. Rubin, H. Taylor (1979). Choosability in Graphs, Congressus Numerantium. 26. pp. 125-157.

[2] C. Thomassen (1994). Every Planar Graph Is 5-Choosable, Journal of Combinatorial Theory, B, vol 62, pp. 180-181.

[3] D.B. West (2001). Introduction to Graph Theory, 2ed.