Interval-Permutation Segment Graphs Overview

Geometric Models of Graphs

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Outline

Outline

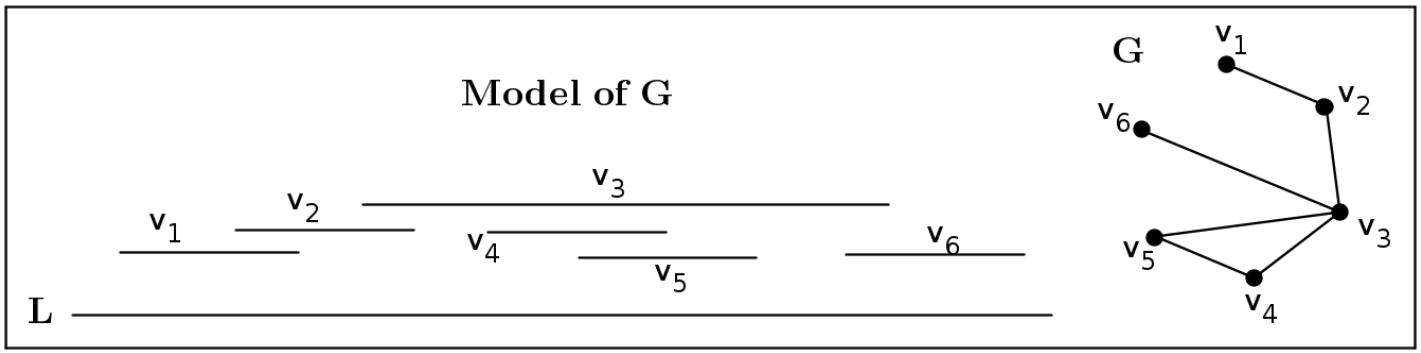
Graphs and Geometry

- **Graphs** show a particular way in which a **set of objects** are **related**.
- It's intuitive to visualize this set of objects as **points**, and their relationships as **lines** intersecting these points.
- Many geometries involve a system of points and lines with various rules constraining their intersection.
- So it is also intuitive to model certain graph classes explicitly in terms of geometric systems.

Outline

An **Interval graph** G can be modelled by a set of line segments $v_1, ..., v_n$, called **interval segments** constrained by the following rules:

1. They are all parallel to a reference line L
2. Interval segments **intersect** if and only if they overlap relative to L.



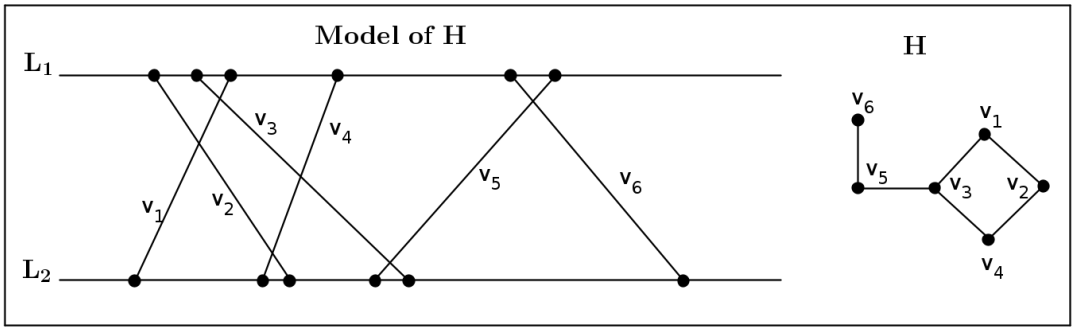
- (**Vertices**) Each interval segment $v_1, ..., v_n$ of the model corresponds to a vertex in G
- (**Edges**) Vertices are adjacent if and only if their interval segments in the geometric model intersect.

Outline

Definition

A **Permutation graph** H can be modelled by a set of line segments $v_1, ..., v_n$, called **permutation segments** constrained by the following rules:

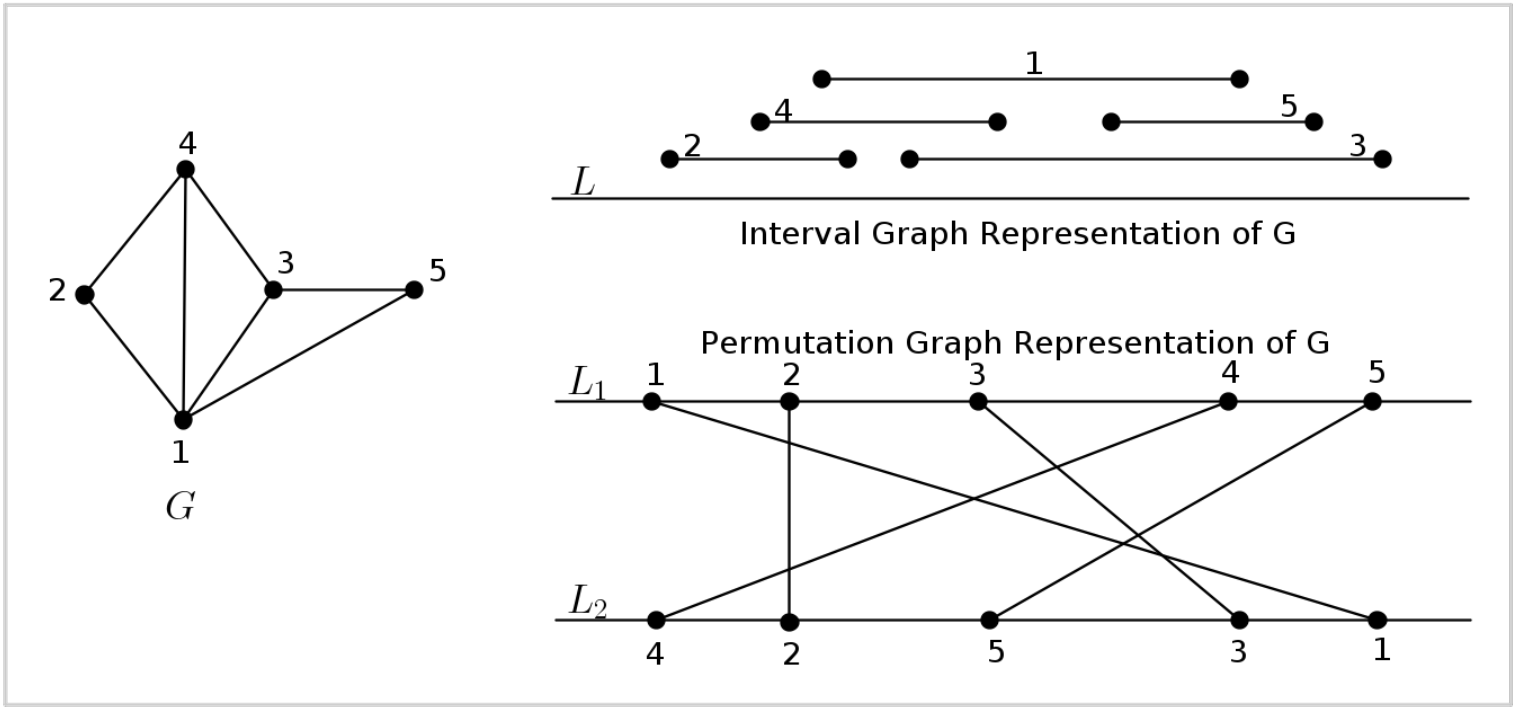
1. End points of permutation segments are on two reference lines L_1 and L_2
2. They intersect if and only if they cross each other between L_1 and L_2.



- (**Vertices**) Each permutation segment $v_1, ..., v_n$ of the model corresponds to a vertex in H
- (**Edges**) Vertices are adjacent if and only if their permutation segments in the geometric model intersect.

Examples

Here is an example of a graph that is both an interval and permutation graph:



Outline

Motivation for Paper

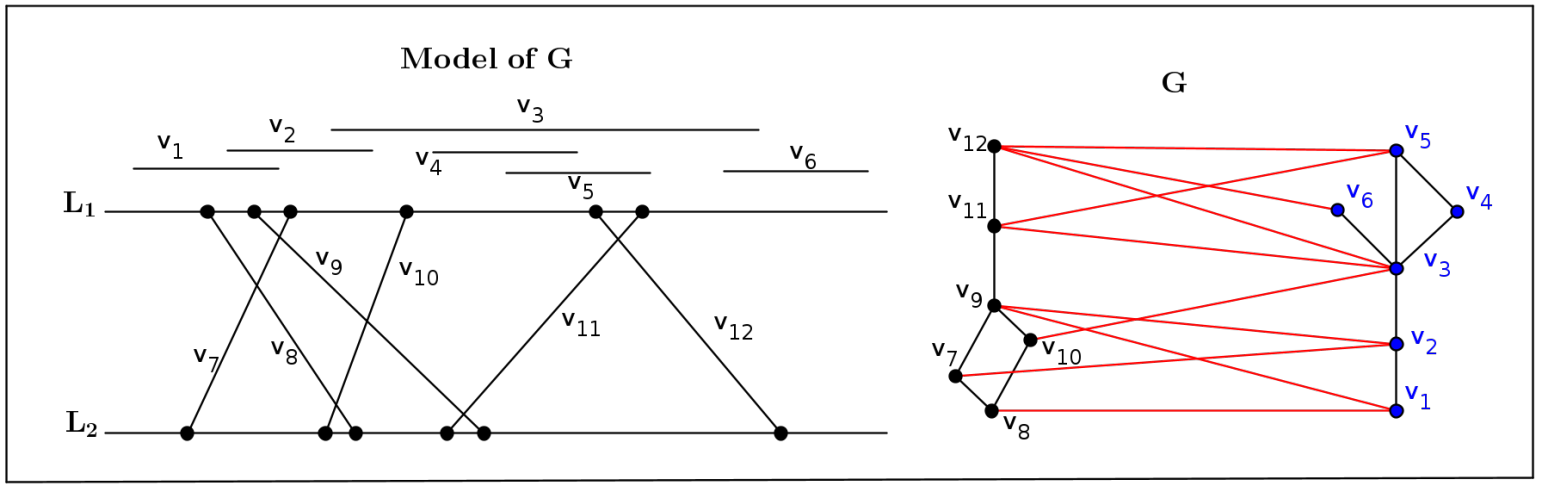
What happens when we join the geometric models of interval and permutation graphs together?

Paper: *Interval-Permutation Segment Graphs* by Z. Joveski and J.P. Spinrad

- The paper answers this question:
  - Propose two new related geometric models that are built from this joining
  - Explore the classes of graphs represented by these two models
  - Show that one is a proper subclass of the other
  - Demonstrate optimization algorithms using the new models.

Intuition Slide

What happens when we join the geometric models of interval and permutation graphs together?



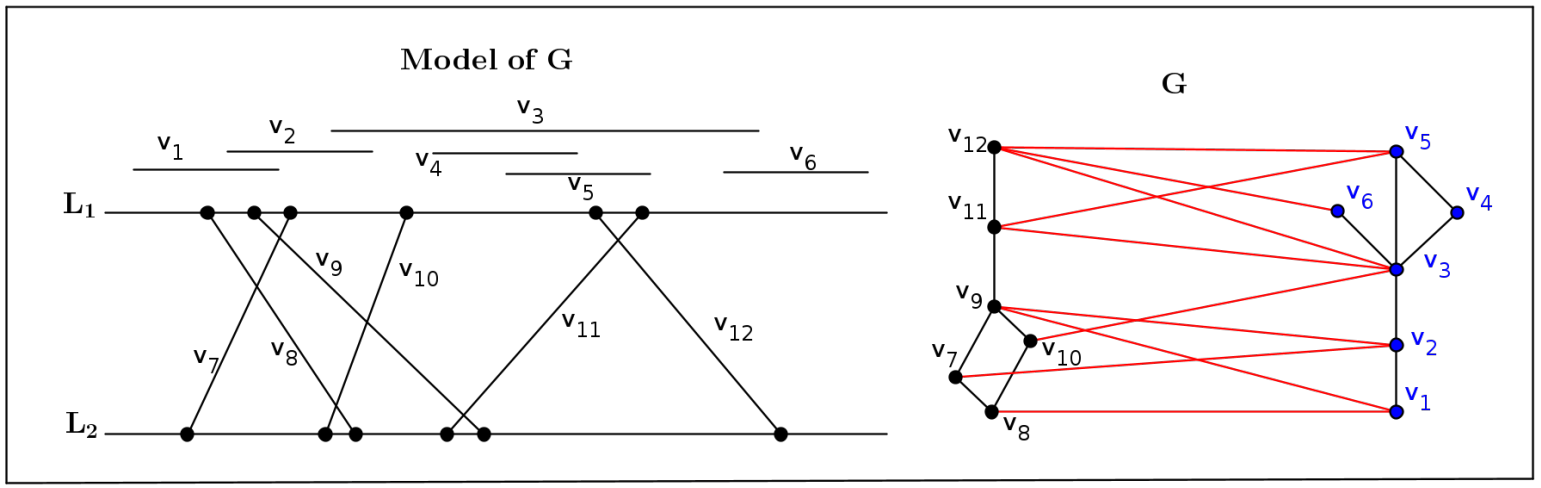
Definition of IP-SEG and IP-SEG* Graphs

- Let I be a set of interval segments, and P be a set of permutation segments.
- Let G be the intersection graph of $I \cup P$. That is, G has a vertex corresponding to each interval and permutation segment; and, vertices are adjacent when:
  - Interval segments overlap one another
  - Permutation segments cross one another
  - Interval segments overlap an endpoint of a permutation segment
- G is called an **interval-permutation-segment (IP-SEG)** graph.
- If all interval segments are on the same reference line, then G is called an **IP-SEG*** graph.

Outline

Intuition Slide (Again)

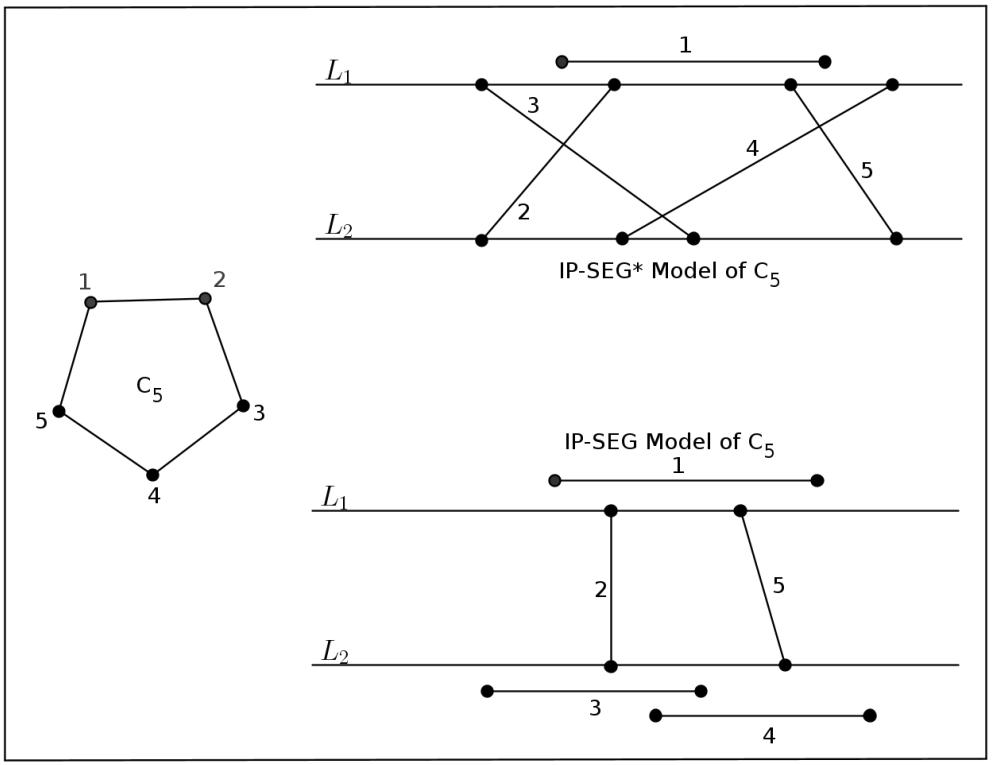
This is an IP-SEG* graph because it has an IP-SEG* model.



That is, it's interval segments are all relative to one reference line (L_1).

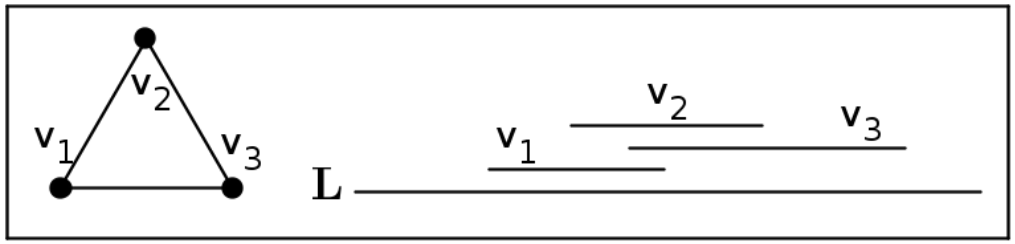
The 5 Cycle

Some graphs have both an IP-SEG* model as well as an IP-SEG model.

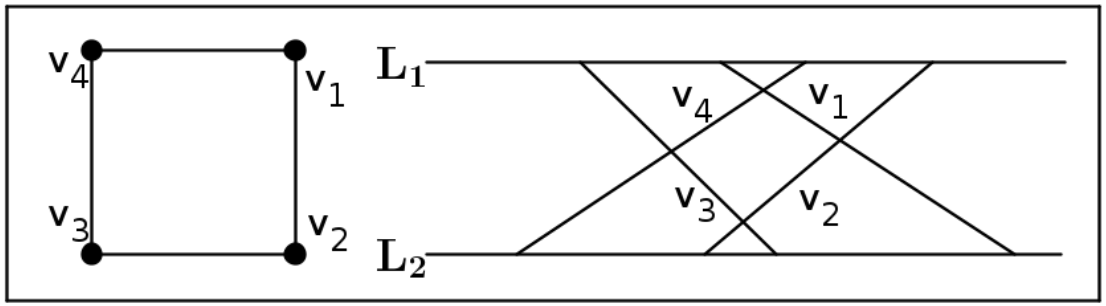


But C_5 is Neither an Interval nor Permutation Graph!

- Interval graphs cannot contain C_n where $n > 3$.



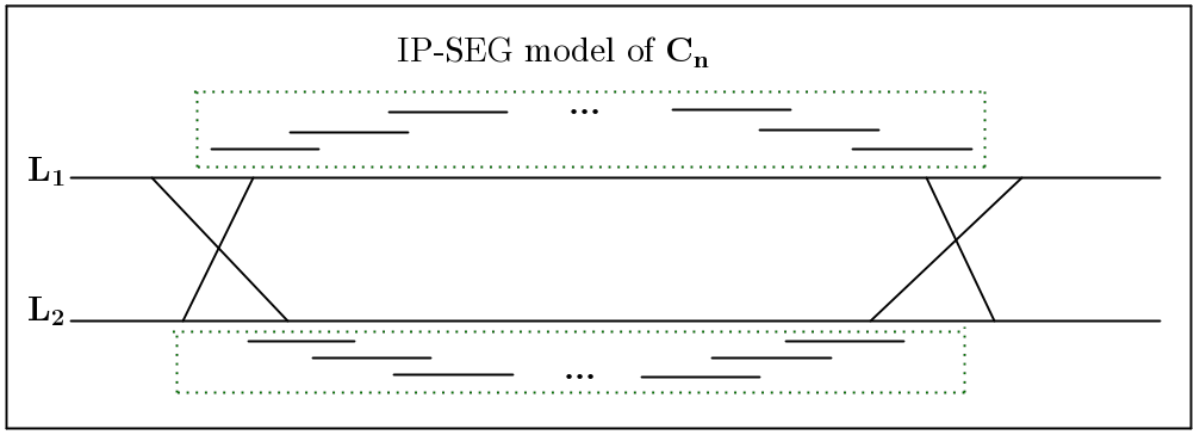
- Permutation graphs cannot contain C_n where $n > 4$.



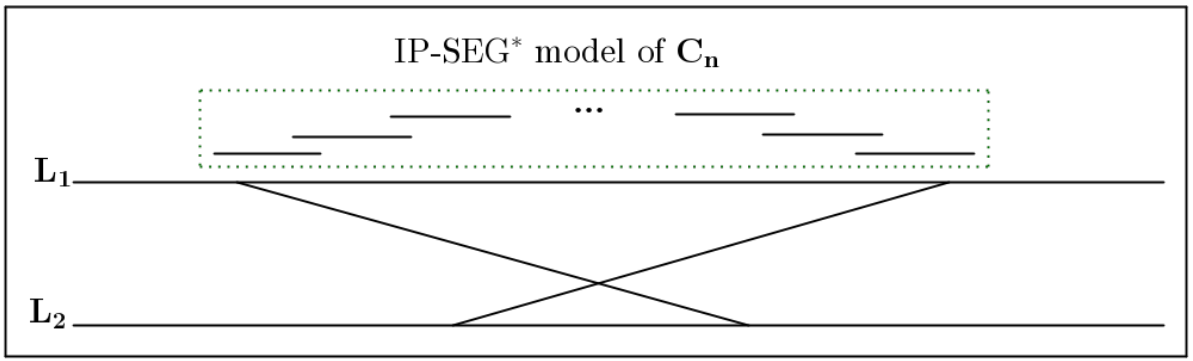
- However, as we've already seen by example, IP-SEG and IP-SEG* graphs can.

IP-SEG Graphs and Chordless Cycles

IP-SEG graphs include all chordless cycles:



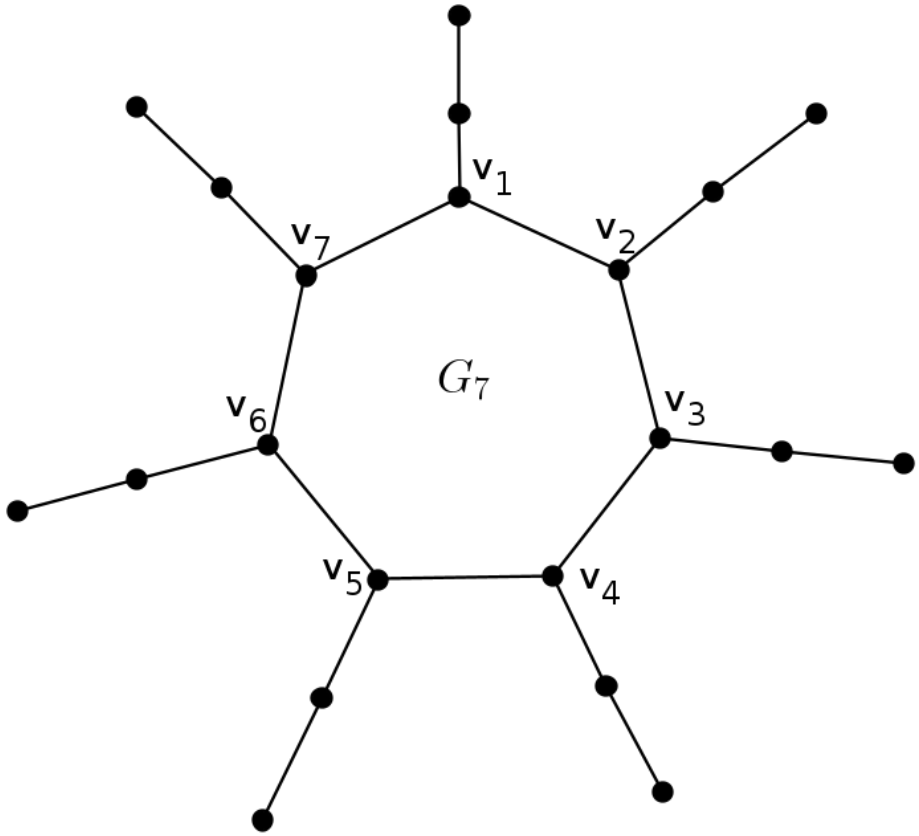
As do IP-SEG* graphs:



IP-SEG* $\subset$ IP-SEG

Not all IP-SEG graphs are IP-SEG* graphs. Consider the following graph class G_n.

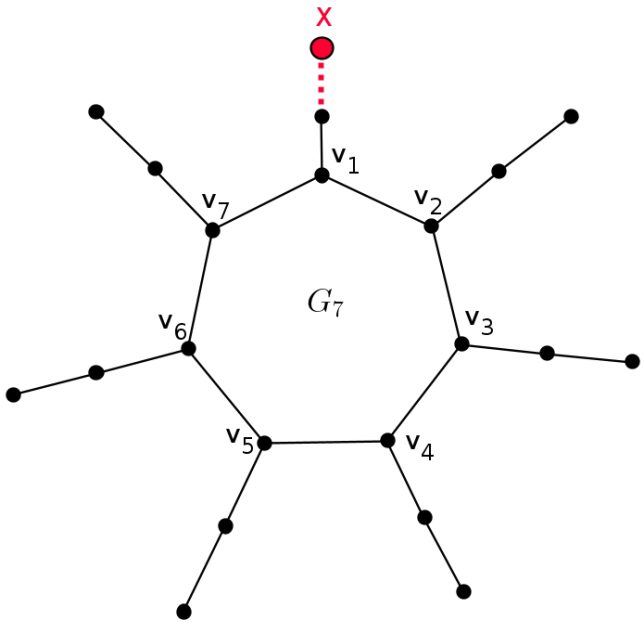
- G_n is a chordless cycle C_n but with a K_2 appended to each vertex in the cycle.

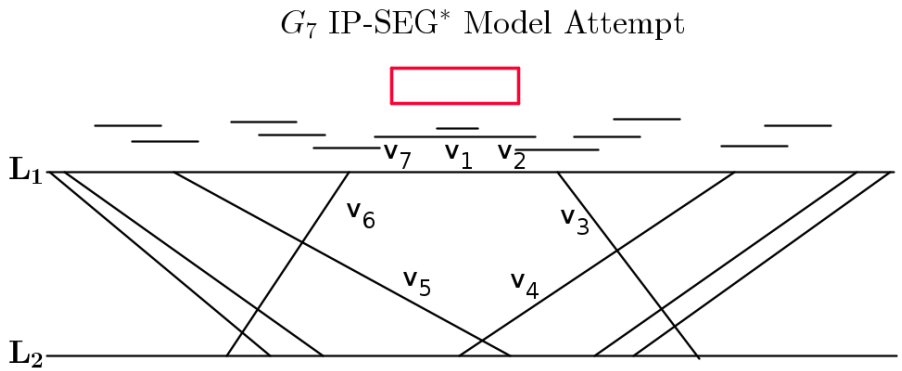


Counter Example

Suppose there did exist an IP-SEG* model for G_7.

- We would be able to associate interval and permutation segments to all but 1 vertex, X.





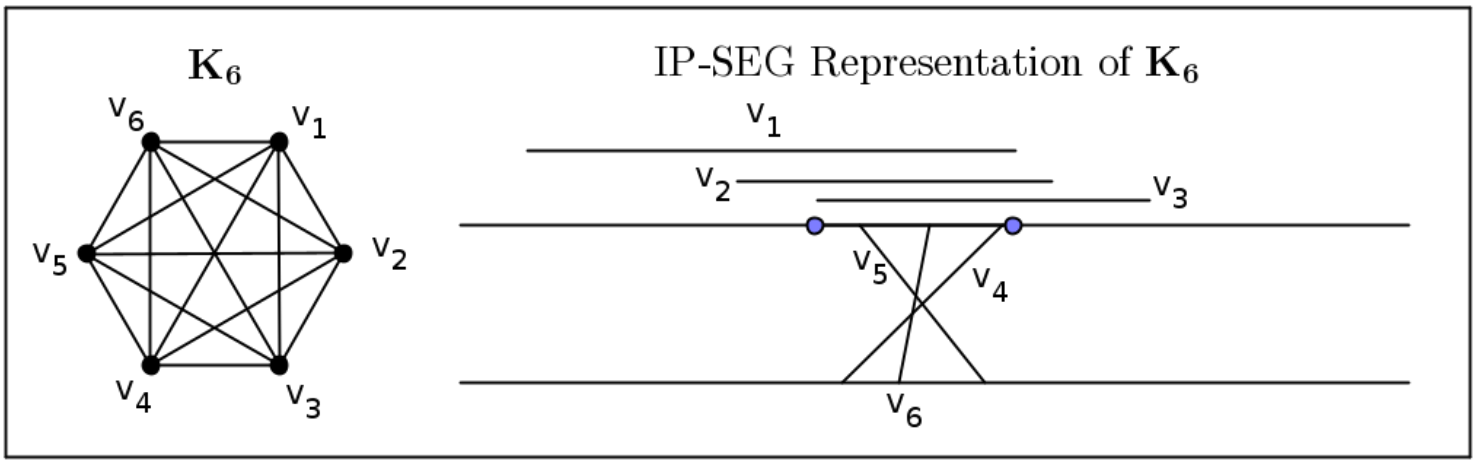
Therefore, IP-SEG* $\subset$ IP-SEG.

Outline

Clique Segment

What do cliques look like in an IP-SEG model?

- The segments associated with the clique vertices are all contained within a pair of endpoints potentially from two different interval segments.



Clique Algorithm Preamble

- The authors present a simple algorithm for finding a maximum clique of a graph with a given IP-SEG model.
- This algorithm is meant to be a demonstration of how the IP-SEG model can be used in optimization.
- It isn't meant to be particularly efficient or practical.

Clique Algorithm

1. Find the maximum clique of the graph induced by all permutation segments. Call it C_{max}.

2. For each pair of endpoints s_1, s_2 on each reference line, do the following:

   2.1 Place all interval segments containing the entire interval $[s_1, s_2]$, into C_{int}.

   2.2 Find max clique formed by permutation segments with an endpoint in $[s_1, s_2]$, and place it into C_{prm}.

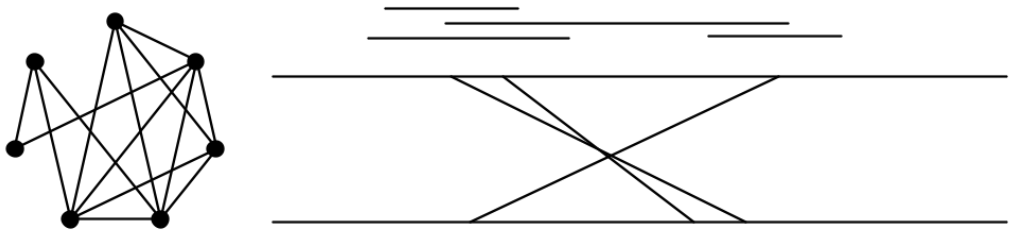
   2.3 Set $C = C_{int} \cup C_{prm}$.

   2.4 If $|C| > |C_{max}|$, set $C_{max} = C$.

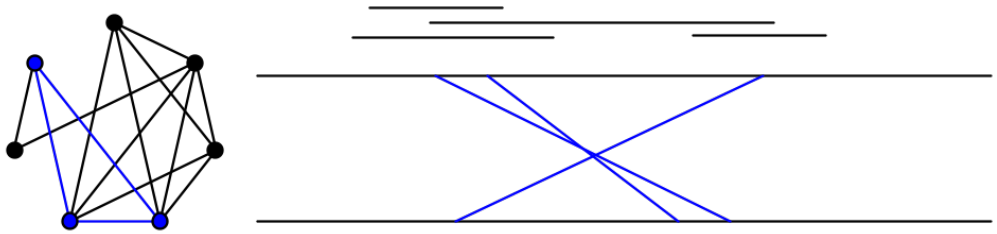
3. Return C_{max}.

Maximum Clique Example

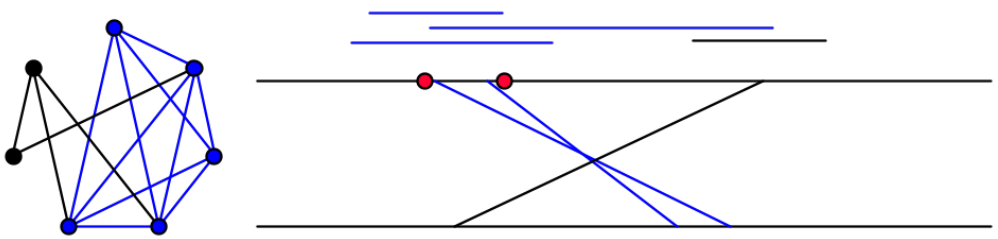
Finding a maximum clique of the following graph:



Step 1: Find a maximum clique involving just the permutation segments



Step 2: Find a pair of endpoints on a reference line intersecting the maximum number of segments:



Summary

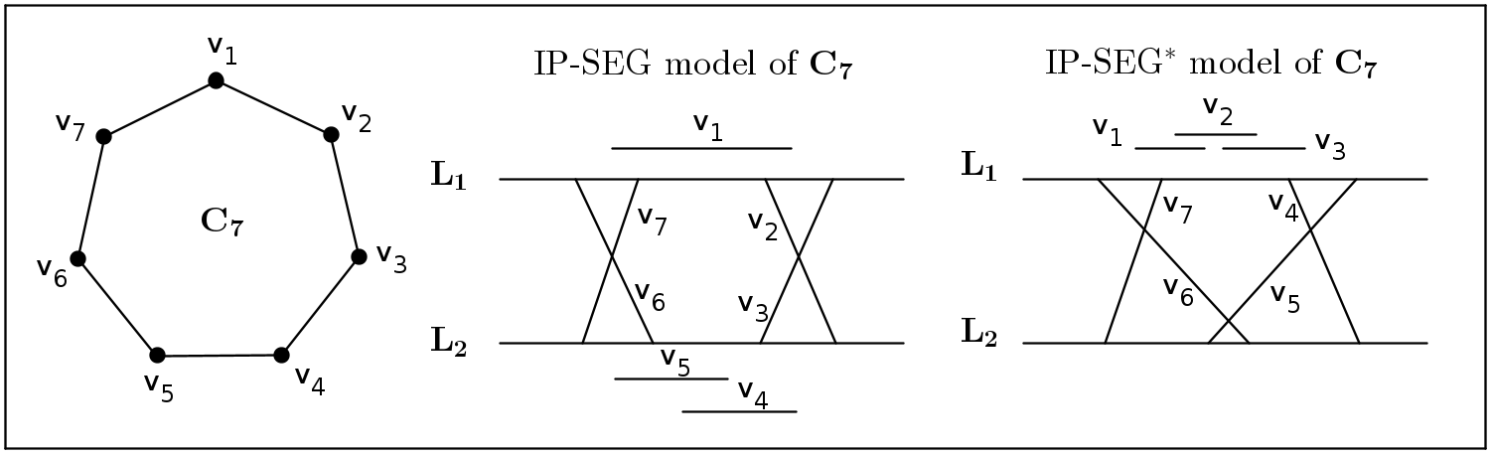
What did we talk about?

1. Using geometric intersection models to represent graphs
2. IP-SEG and IP-SEG* graphs as generalizations of interval and permutation graphs
3. IP-SEG graphs include graphs that are neither interval nor permutation graphs
4. Forbidden subgraphs: $G_7 \in$ IP-SEG, but $\notin$ IP-SEG*, and $G_n \notin$ IP-SEG, $n \geq 9$
5. Clique problem using an IP-SEG model

Outro

Thank you for your time!

- I hope you enjoyed the talk and that you now have a better intuition for how to relate geometric systems to graphs.



Void Slide

Interval Arc Definition

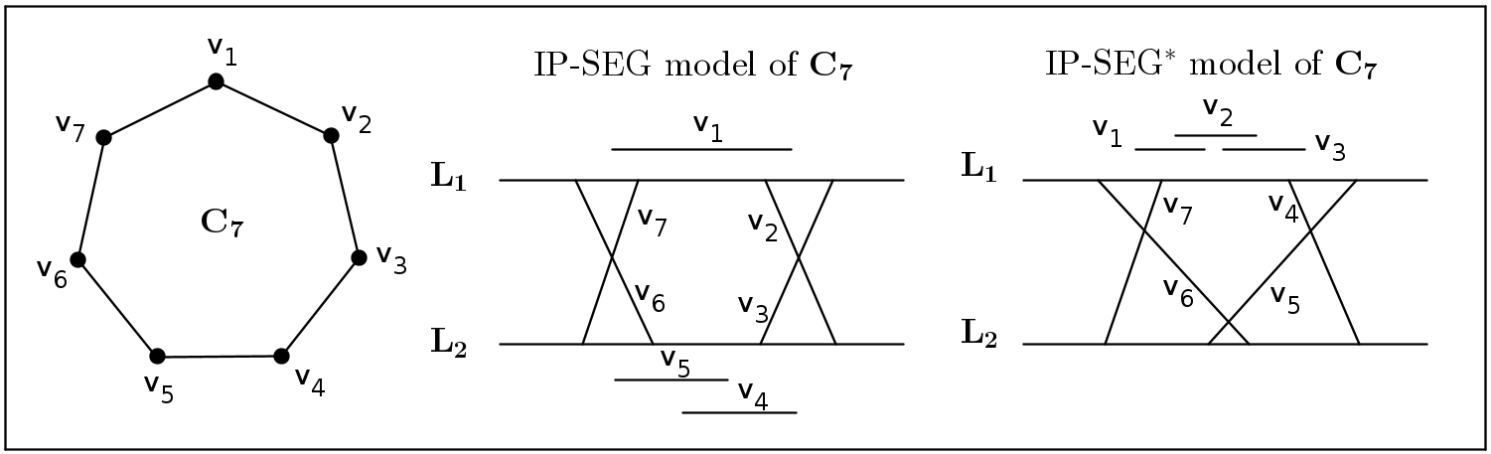
- IP-SEG and IP-SEG* representations of C_n always involve multiple interval segments and permutation segments.
- We can get a more global picture of our models if we make a concept that groups intersecting segments by type:
  - **Interval arc:** sequence of intersecting interval segments.
  - **Permutation arc:** sequence of intersecting permutation segments.

Every C_n, with $n > 4$ can be represented as an alternating sequence of interval and permutation arcs.

Interval Arc Example: C_7

- The IP-SEG model of C_7 has four arcs:
  $I_1 = (v_1), P_1 = (v_2, v_3), I_2 = (v_4, v_5), P_3 = (v_6, v_7)$
  - **two** interval arcs (on different lines) and **two** permutation arcs

- The IP-SEG* model of C_7 has two arcs:
  $I_1 = (v_1, v_2, v_3)$, and $P_1 = (v_4, v_5, v_6, v_7)$
  - **one** interval arc and **one** permutation arc

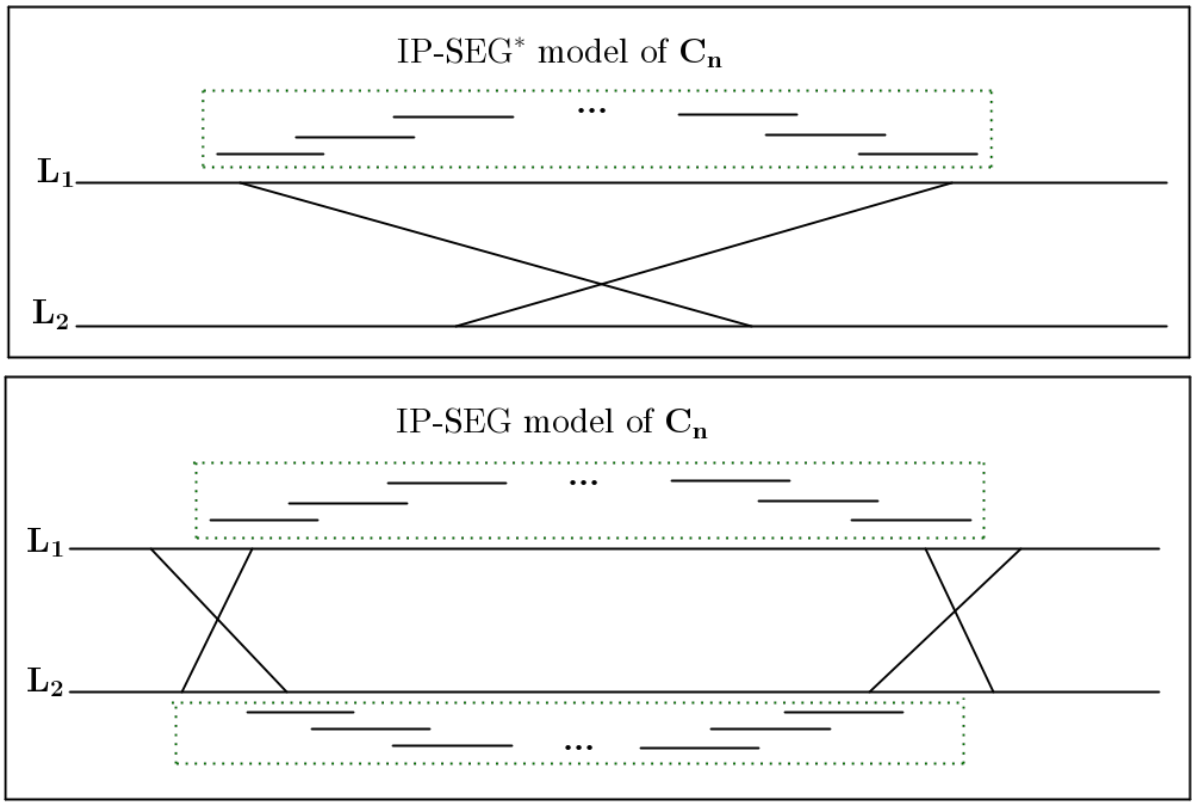


Theoretical Results: Characterizing C_n Models

- The theoretical results characterize the IP-SEG and IP-SEG* representations of C_n.
- The authors prove the validity of a technique involving interval arcs.
- Then they use this technique to characterize exactly what the IP-SEG and IP-SEG* models are for C_n.

Technique

Lemma 4.1: C_n has an IP-SEG representation if and only if the cycle with the same interval arcs as C_n shrunk down to size 1 has an IP-SEG representation.



Characterizations

- **Lemma 4.2:** An IP-SEG* model of chordless cycle must consist of exactly one interval arc

- **Lemma 4.3:** An IP-SEG model of a chordless cycle must either consist of
  - exactly one interval arc, or
  - exactly two interval arcs, each positioned on different reference lines L_1 and L_2.

Lemma 4.3 suggests that there are graphs that have IP-SEG models but NOT IP-SEG*.