Combinatorics and Music

Tonnetz Systems

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Outline

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1. Fundamental concepts relating pitch classes/notes to sets
2. Integer compositions as things that express the "shape" of harmonies and sets generally.
3. Can define Tonnetz set systems in terms of these integer compositions
4. Mathematical properties and questions regarding Tonnetz systems

Outline

1. Introduction
   - Overview
   - Pitch Classes
   - Integer Compositions
2. Tonnetz Systems
   - Definitions and Basic Properties
   - Music Examples
   - Questions/Conjectures

Pitch Classes

- Let $[n] = \{0, 1, ..., n-1\}$.
- When $n = 12$, there are **two ways** each element of [12] can represent pitch classes cyclically:

Incrementing by **perfect fifth intervals**:

0	1	2	3	4	5	6	7	8	9	10	11
C	G	D	A	E	B	$F\#$	$C\#$	Ab	Eb	Bb	F

Incrementing by **semitone intervals**:

0	1	2	3	4	5	6	7	8	9	10	11
C	$C\#$	D	Eb	E	F	$F\#$	G	Ab	A	Bb	B

Harmonies

- Let t be a subset of $[n]$ with size k. Then t is called a **k-subset** of $[n]$.
- When $n = 12$, t represents a **harmony** with $1 \leq k \leq 12$ pitch classes.
- When $1 < k < 5$, t is called a **chord**, and when $5 \leq k \leq 12$, t is called a **scale**.
- *Example:* the C-major scale represented in semitone and fifth ordering, respectively.

$$\begin{cases} \{C, D, E, F, G, A, B\} \to \{0, 2, 4, 5, 7, 9, 11\} \\ \{F, C, G, D, A, E, B\} \to \{11, 0, 1, 2, 3, 4, 5\} \end{cases}$$

Studying Harmonies

A useful way to study harmonies as sets is to consider the consecutive differences between their pitch classes, these are called **intervals**:

1. Order them cyclically (either semitone or fifth ordering). Eg:

$$\text{Fifth Ordering: } \{C, G, E\} \to \{0, 1, 4\}$$

$$\text{Semitone Ordering: } \{C, E, G\} \to \{0, 4, 7\}$$

2. Consider the consecutive differences modulo 12 between their elements. Eg:

$$\text{Fifth Ordering: } \{0, 1, 4\} \to (1, 3, 8)$$

3. The resulting object is a **rotation invariant composition** of $n = 12$. We want rotation invariance because we consider harmony inversions to be equivalent. Eg: $(C, E, G) = (E, G, C)$.

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Why Compositions are Interesting

- Rotation invariant compositions of 12 are interesting musically, because they combine the intervals between the pitch classes in a harmony into one object.
- This allows us to classify harmonies by their shape, or more formally **harmonic species**.

$$\text{The Major Triads} = \begin{cases} \{0,1,4\},\{1,2,5\},\{2,3,6\}, \\ \{3,4,7\},\{4,5,8\},\{5,6,9\}, \\ \{6,7,10\},\{7,8,11\},\{8,9,0\}, \\ \{9,10,1\},\{10,11,2\},\{11,0,3\}. \end{cases} \rightarrow (1,3,8)$$

Compositions are Interesting Continued

- These compositions give us a concise way to describe the combinatorial shape of harmonies.
- Now, we can investigate patterns involving these shapes
- Ultimately, the hope is that some of such investigations inform new ways of thinking about music harmony.

More Definitions Relating to Compositions

Let C be a rotation invariant n-composition. Then

- Define $R(C)$ as C but with its elements in the reversed cyclic ordering. $R(C)$ is called the **reversal**, or **flip**, of C.
- If $C = R(C)$, then C is **achiral.**
  Eg: $R(1,2,1,3) = (3,1,2,1) = (1,2,1,3)$
- If $C \neq R(C)$, then C is **chiral.**
  Eg: $R(2,4,6) = (6,4,2) = (2,6,4) \neq (2,4,6)$
- $w(C)$ is called the **weight** of C, and it is the sum of all its elements. It is always true that $w(C) = n$.

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Tonnetz System Definition

- The **tonnetz system** $T_{n,k}(C)$ is the k-subset family of $[n]$ that includes all k-subsets whose consecutive differences (modulo n) are given by the rotation invariant integer compositions C and $R(C)$. Can also denote simply as $T(C)$ since $k = |C|$ and $n = w(C)$.
- **Examples:**

$$T_{6,3}(123) = \begin{cases} \{013\}, \{124\}, \{235\}, \{340\}, \{451\}, \{502\}, \\ \{053\}, \{104\}, \{215\}, \{320\}, \{431\}, \{542\} \end{cases}$$

$$T_{5,3}(122) = \{\{013\}, \{124\}, \{230\}, \{341\}, \{402\}\}$$

$$T_{6,4}(1212) = \{\{0134\}, \{1245\}, \{2350\}\}$$

Size

The **size** of a Tonnetz system is given by:

$$|T_{n,k}(C)| = \frac{cn}{r} \tag{1}$$

where $C = x^r$ for some possibly smaller composition x and integer r; and $c = 1$ if C is achiral, $c = 2$ otherwise.
Note: $r|n$ because $n = r \cdot w(x)$.

$(\mathbf{r}, \mathbf{c})$

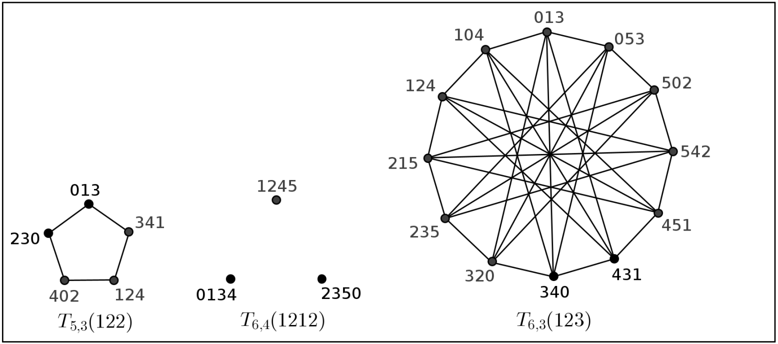
$(1, 1) : T_{5,3}(122) = \{\{013\}, \{124\}, \{230\}, \{341\}, \{402\}\}$

$(2, 1) : T_{6,4}(1212) = \{\{0134\}, \{1245\}, \{2350\}\}$

$$(1, 2) : T_{6,3}(123) = \begin{cases} \{013\}, \{124\}, \{235\}, \{340\}, \{451\}, \{502\}, \\ \{053\}, \{104\}, \{215\}, \{320\}, \{431\}, \{542\} \end{cases}$$

Adjacency

We can define a **graph** for each $T_{n,k}(C)$ with k-subsets as vertices, which are adjacent if and only if they share all but i elements.
Here, $i = 1$.



Tonnetz Systems and Johnson Graphs

The graph $T_{n,k,i}(C)$ is an induced subgraph of the **Johnson graph** $J_{n,k,i}$:

- In $J_{n,k,i}$ all k-subsets are included, and as with $T_{n,k,i}(C)$, the k-subsets that share all but i elements are adjacent.
- Note that $|J_{n,k,i}| = \binom{n}{k}$ whereas $|T_{n,k,i}(C)| = \frac{cn}{r}$
- So, C dramatically constrains the number of k-subsets allowed in $T_{n,k}(C)$.

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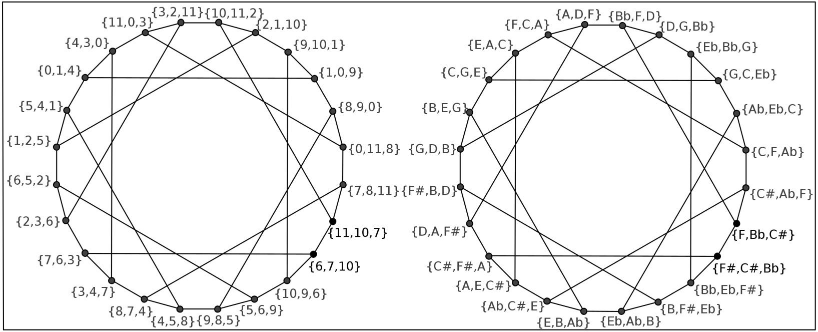
Music Example

The subsets in $T_{12,3,1}(138)$ correspond to the major and minor triads in 12-tone equal temperament harmony:



Other Triad Tonnetz Systems

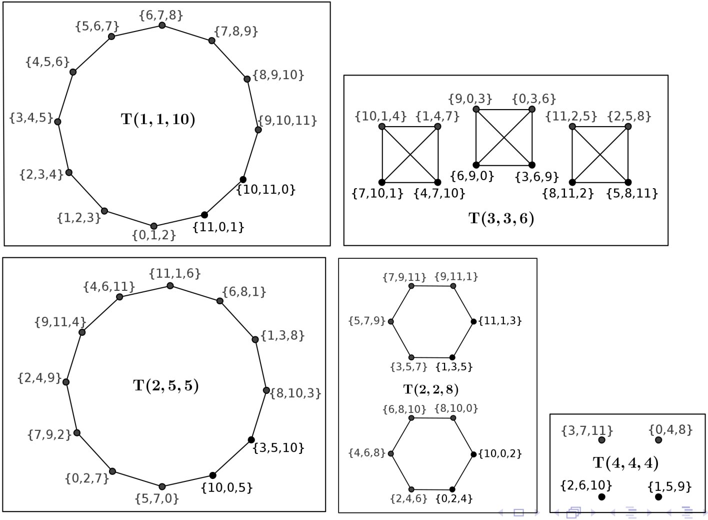
What about the rest of the triad tonnetz systems? There are 11 others! How big are they? Remember $|T| = \frac{nc}{r}$ and $n = 12, k = 3$.

$$\text{Achiral: } \begin{cases} |T(1,1,10)|, |T(228)|, \\ |T(255)|, |T(336)| \end{cases} = \frac{(12)(1)}{(1)} = 12,$$

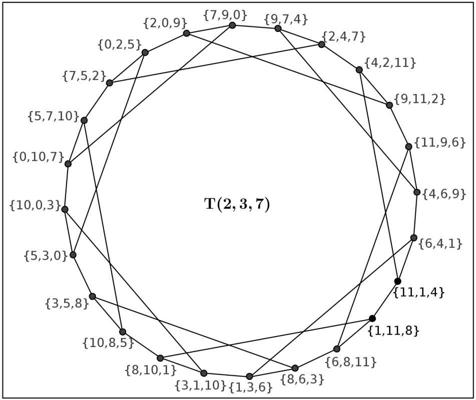
$$\text{Chiral: } \begin{cases} |T(129)|, |T(147)|, |T(156)|, \\ |T(237)|, |T(246)|, |T(345)|. \end{cases} = \frac{(12)(2)}{(1)} = 24,$$

$$\text{Achiral \& Repeated Subsequence: } |T(444)| = \frac{(12)(1)}{(3)} = 4.$$

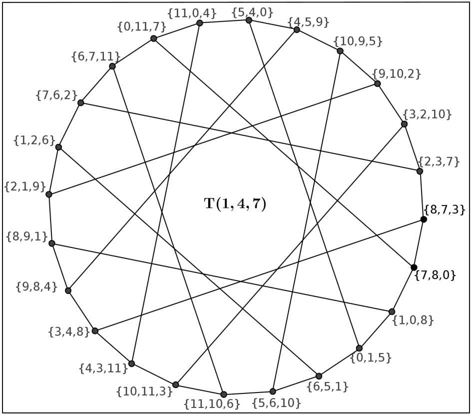
Achiral $T_{12,3,1}$ Graphs



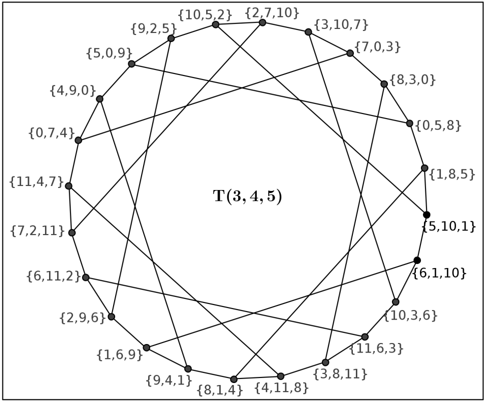
Chiral $T_{12,3,1}(2,3,7)$ Graph



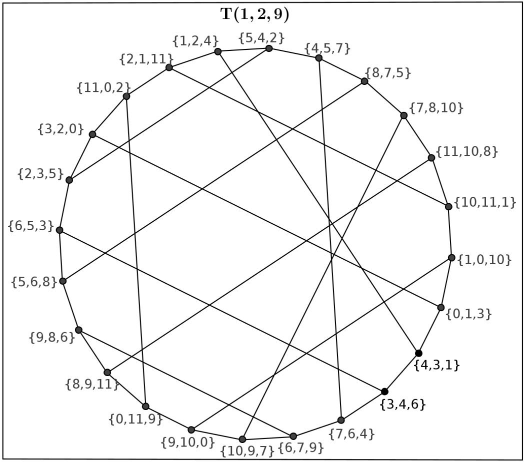
Chiral $T_{12,3,1}(1,4,7)$ Graph



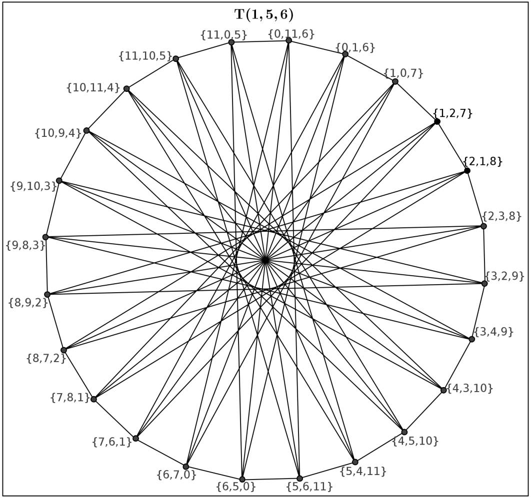
Chiral $T_{12,3,1}(3,4,5)$ Graph



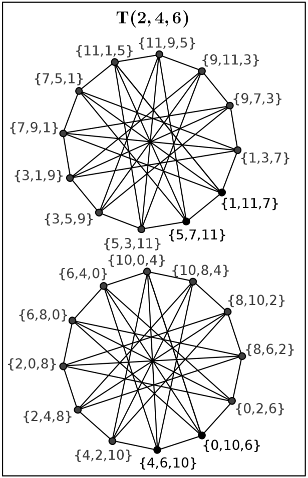
Chiral $T_{12,3,1}(1,2,9)$ Graph



Chiral $T_{12,3,1}(1,5,6)$ Graph



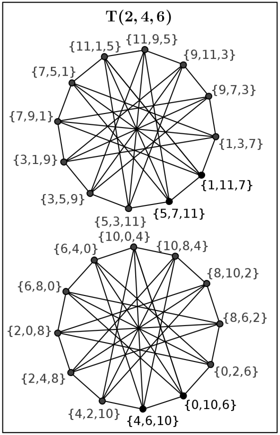
Chiral $T_{12,3,1}(2,4,6)$ Graph

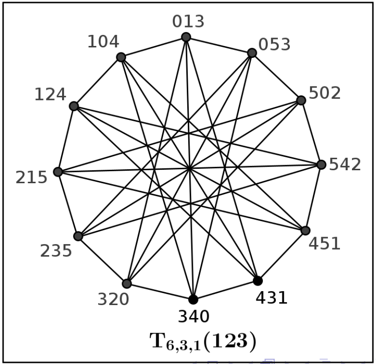


Outline

Relationship: $T_{12,3,1}(2,4,6)$ and $T_{6,3,1}(1,2,3)$

The components in $T_{12,3,1}(2,4,6)$ are isomorphic to $T_{6,3,1}(1,2,3)$:





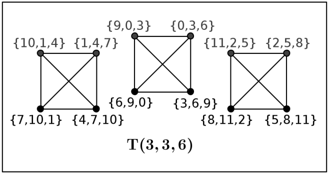
Primitive Component Conjecture

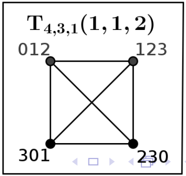
Conjecture 1:
If $n = ts$ for some positive integers s and t, and t divides each element in C, then

$$T_{n,k,i}(tC) \cong tT_{s,k,i}(C) \tag{2}$$

That is, $T_{n,k,i}(tC)$ is isomorphic to t copies of $T_{s,k,i}(C)$. Also seems true that $T(tC) \cong tT(C)$.

Example: $T_{12,3,1}(3,3,6) \cong 3T_{4,3,1}(1,1,2)$





Relating Composition Transformations to Tonnetz Graphs

Relationships between compositions can correspond to relationships between their induced tonnetz systems and graphs.

- We will look at composition **complement** and **concatenation**
- These seem to suggest corresponding isomorphisms between tonnetz system graphs

Composition Complementation

Let $C = (c_1, c_2, ..., c_k)$ be a composition in some rotation, then we define the **complement** of C as $\overline{C} = b^{-1}(s(b(C)))$ where

1. *To binary:* $b(C) = (10^{c_1-1}, 10^{c_2-1}, ..., 10^{c_k-1})$
2. *Swap bits:* $s(C)$ swaps 1s for 0s and 0s for 1s, and
3. *To Base Ten:* $b^{-1}(C)$ converts every subsequence of the form (10^{j-1}) into (j)

For example: $\overline{(138)} = (121111113)$

$$\begin{aligned}\overline{(1,3,8)} &= b^{-1}(s(b((1,3,8)))) \\ &= b^{-1}(s(110010000000)) = b^{-1}(001101111111) \\ &= (1,2,1,1,1,1,1,1,3) = (111111312)\end{aligned}$$

Tonnetz Complement

Define $\overline{T(C)}$ to be the set system derived by taking the complement of every set in $T(C)$ relative to $[n]$.

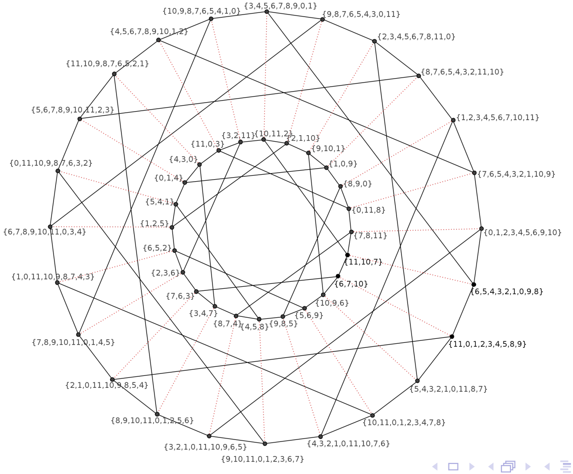
$\overline{T(C)}$ is called the **tonnetz system complement** of $T(C)$.

Observation:

$$\overline{T_{n,k}(C)} = T_{n,n-k}(\overline{C})$$

Tonnetz Complement Example

$$T_{12,3,1}(138) \cong T_{12,9,1}(111111312)$$



Complement Conjecture

Conjecture 2: If $T_{n,k}(C)$ is a Tonnetz system and $i \leq min\{k, n-k\}$, then

$$T_{n,k,i}(C) \cong \overline{T_{n,k,i}(C)}$$

If true, then we automatically learn a lot about $\overline{T(C)}$ just by studying $T(C)$.

Concatenation

Let $T_{n,k}(x)$ be a tonnetz system and r a positive integer, then we can use concatenation on compositions to define a transformation G (for **grow**) on Tonnetz systems as follows:

$$G(T_{n,k}(x), r) = T_{rn,rk}(x^r) \tag{3}$$

For each set S in $T_{n,k}(x)$, we define $S' = S + n$, and include S' into S. We apply this process $r - 1$ times.
For example, $G(T_{6,3}(123), 2) = T_{12,6}(123123)$

$G(T_{6,3,1}(123), 2) = T_{12,6,2}(123123)$

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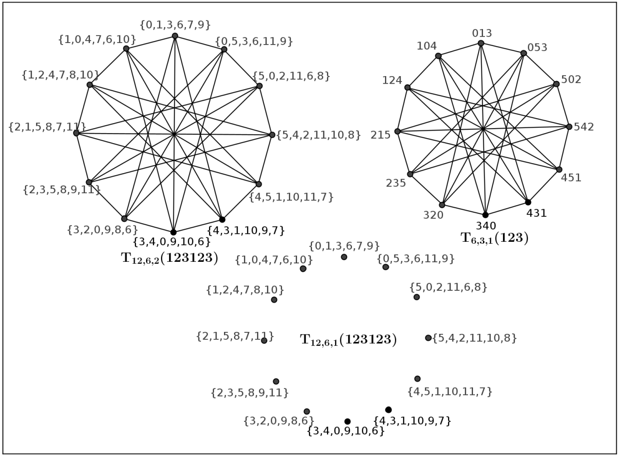


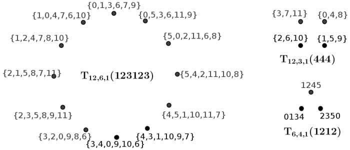
Generalizing to Conjectures

Conjecture 3: Let $n, k, i \in \mathbb{Z}^+$ such that $1 \leq i \leq k \leq n$. Then $\forall r \in \mathbb{Z}^+$,

$$T_{n,k,i}(C) \cong T_{rn,rk,ri}(C^r)$$

If true, then can use G to define an ER and study representatives with minimal parameters.

Conjecture 4: Let $T(C)$ be a tonnetz system with $C = x^r$. If $r > 1$, then its Johnson subgraph for $i = 1$ consists entirely of isolated vertices.



If true, then maybe always true when $i < r$?

Summary

- How to think about harmonies in terms of sets
- Using integer compositions to constrain the shape of sets/harmonies
- Introduction to Tonnetz systems and their graphs and my various questions

I hope you enjoyed the talk!

If you have any thoughts, reflections or ideas, let me know!
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EXTRA: Union of Tonnetz Systems

- What do we get when we take the union of Tonnetz systems?
- Sometimes we get cool Johnson subgraphs like the one on the next slide.
- It's not obvious to me what sort of graph we should expect associated with given union of Tonnetz systems.
- However, we do know that the intersection of any pair of tonnetz systems is empty.

Example: $T(1,3,8) \cup T(3,4,5)$ has an associated 48 vertex 7-regular Johnson subgraph.

$T(1,3,8) \cup T(3,4,5)$ Graph