A Project Proposal for a Mathematical Study of Rhythm Tilings

By Tao Gaede
tao.gaede@live.ca

Summary:

The object of interest in this proposal is a two dimensional toroidal binary array constructed from a tiling of cyclic one dimensional binary arrays, here called a *rhythm tiling*. These rhythm tilings are related to a well studied object in the mathematical music theory literature called *tiled rhythm canons* (see [2 - 5]). Both of these objects can be used in music composition to integrate systematically the rhythms of multiple voices in a phrase of music. That is, these objects can be used as tools for orchestration.

Rhythms (defined below) have an interesting property called evenness, which has been defined in multiple ways (see my algorithm below, [8], and [14]). In fact, much work has been done on the optimized case of maximally even rhythms, known as Euclidean rhythms (see [1], [10], [13], [14], and [15]). I am interested in how to extend this evenness notion beyond rhythms to rhythm tilings. Further, I define the notion of a Euclidean rhythm tiling, and propose an investigation of its properties. The main goal of this project is to explore and uncover theoretical results (particularly existence, enumeration, and construction methods) on rhythm tilings. The secondary goal is to explore evenness properties of rhythm tilings.

Outline:

I will begin with constructing the notion of a rhythm with the help of rotationally invariant integer compositions. The rhythm tiling will then be defined, followed by an introduction to evenness of rhythms. Maximally even rhythms, also known as Euclidean rhythms will be introduced, followed by the definition of a Euclidean rhythm tiling.

Definition (Rotational Invariant Integer Compositions):

What follows is a construction from integer partitions:
Let k be a positive integer.
A *partition* of k is a set of $0 < n \leq k$ positive integers that add up to exactly k.
A *composition* of k is like a partition except in a composition, order matters; that is, a composition of k is a sequence of $0 < n \leq k$ positive integers that add up to exactly k.
An object $C = (c_1, ..., c_n)$ is called a *rotationally invariant composition* of k, when $(c_1, ..., c_n) \sim (c_2, ..., c_n, c_1) \sim ... \sim (c_n, c_1, ..., c_{n-1})$, where ~ denotes the rotational equivalence relation on finite sequences. So, ~ partitions the set of compositions of k with length n into equivalence classes. We will call this set $C_{k,n}$, that is, the set of rotationally invariant compositions of k with length n.

Also define $C_k = \bigcup_{n=1}^{k} C_{k,n}$.

$C_{k,n}$ will be used to more concisely describe certain properties of rhythms.
A rhythm can be defined in many ways, one of them is the following.

Definition (Rhythm):

A rhythm is a binary necklace of length k and weight (or number of 1s) equal to n, in which the 1s denote "on" and the 0s denote "off". For each rhythm, the distribution of the 1s can be characterized by an object $C(R)$ from the set $C_{k,n}$, whereby the entries in this object correspond to one plus the number of 0s between neighbouring 1s in R. For example, suppose $R = (0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0)$, then $C(R) = (3\ 1\ 2\ 1\ 6)$, where $C(R) \in C_{13,\,5}$.

An interesting question relevant to music composition is: how can we tile a finite and discrete time axis using a given rhythm R as the tile? Such an object is called a tiled rhythm canon. There is a mathematical and musical literature on rhythm canons, and much of the mathematical work appears to have been done by a prolific Romanian mathematician Dan Tudor Vuza in the early 90s (see [2 - 5]). Since rhythms are rotationally invariant (being necklaces), and since we do not care in what order the tiling of the time axis is completed, tiled rhythm canons are toroidal objects. But we can represent them as binary matrices - keeping in mind that this matrix is only a representative of a class of binary matrices rotationally invariant both horizontally and vertically.

An example:
Given $R = (1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0)$, we can get the following tiled rhythm canon:

$$\begin{bmatrix} R \\ R \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

I am interested in a slightly different rhythm tiling object. The object of interest is one that allows any rhythm to be used in the tiling so long as it satisfies the condition that each column contains exactly one 1. As far as I can tell, this object is not discussed in the rhythm canon literature, which makes sense, because musically, it is not actually a canon. A canon explicitly requires the recurrence of only one rhythm, similarly to how musical rounds repeat the same melody in different voices offset in time. So the object of interest should not be considered a special case of a rhythm canon. Let's call it simply a *rhythm tiling*.

Definition (Rhythm Tiling):

A rhythm tiling RT is a binary toroidal m by k array such that each row i is a rhythm R_i of length k tiled so that the sum of entries in each column is exactly 1. Symbolically, this means the following:

$$\begin{bmatrix} R_0 \\ \vdots \\ R_{m-1} \end{bmatrix} \xrightarrow{Tiling} \begin{bmatrix} R_{0,\,0} & \cdots & R_{0,\,k-1} \\ \vdots & \ddots & \vdots \\ R_{m-1,\,0} & \cdots & R_{m-1,\,k-1} \end{bmatrix}, \text{ whereby } \forall\, j \in [0, k-1],\ \sum_{i=0}^{m-1} R_{i,\,j} = 1.$$

Recall that each rhythm R_i has a weight, call it n_i. Then a rhythm tiling has an associated positive integer necklace $\boldsymbol{n} = (n_0, \ldots, n_{m-1})$, which we will call its *weight necklace.*

For example, the following is a rhythm tiling:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

it has parameters: $m = 4$, $k = 10$. Each rhythm is different, and the weight necklace is $\boldsymbol{n} = (3, 3, 2, 2)$

Brief Note: It is probably okay to relax the row order constraint here in the definition, but row ordering might be important for defining a construction algorithm. Likely, many rhythm tiling construction algorithms would involve beginning with a certain type of rhythm, and then the choice of proceeding rhythms would depend on previously tiled ones. So, row ordering may be helpful for rhythm tiling construction.

On Evenness of Rhythms:

Rhythms have an evenness property to them, which tells us how evenly distributed the 1s are amongst the 0s (if you'll forgive the tautology). There are different evenness measures in the literature, but for our purposes here, we will use one I developed for my work on applying $C_{k,\, n}$ to musical harmonies.

My Evenness Sorting Approach:

Definition (Minimum Sum Function):

For every $i \in [0, n-2]$, define a function $S_i : C_{k,\, n} \to \{1, ..., k\}^i$ where

$$S_i(C(R)) = \min\left\{\sum_{j=l}^{l+i} C(R)_j,\ l \in [0, ..., n-1]\right\}.$$

That is, $S_i(C(R))$ is the minimum sum of the length i proper subsequences of $C(R)$.

Definition (Minimum Sum Sequence):

Define the sequence

$$S(C(R)) = \left(S_{n-2}(C(R)), ..., S_0(C(R))\right).$$

Such a sequence will be referred to both the *sum sequence* of $C(R)$, as well as the *sum sequence of* R. $S(C(R))$ is monotone descending.

Sorting:

So, for every $C \in C_{k,\, n}$, there is a sum sequence $S(C)$. Now define the sequence of all sum sequences for elements in $C_{k,\, n}$, call it E, and sort its contents lexicographically.

Given this sorted E sequence, apply $S^{-1}(S(C(R))) = C(R)$ to each element in E to get the set of $C_{k,\, n}$ sorted by evenness (from least even to highest). To get the set of corresponding rhythms sorted by evenness, further apply $C^{-1}(C(R)) = R$ to each element of E.

For Example:

Below are the rhythms of length $k = 12$ and weight $n = 3$ ordered from least even at the top, to most even at the bottom, according to the evenness sorting algorithm described above:

Even#	*S*	$C_{12,3}$			*R*		
1	(2, 1)	-	(1 1 10)	-	-	(1 1 1 0 0 0 0 0 0 0 0 0)	-
2	(3, 1)	(1 9 2)	-	(1 2 9)	(1 1 0 0 0 0 0 0 0 0 1 0)	-	(1 1 0 1 0 0 0 0 0 0 0 0)
3	(4, 1)	(1 8 3)	-	(1 3 8)	(1 1 0 0 0 0 0 0 0 1 0 0)	-	(1 1 0 0 1 0 0 0 0 0 0 0)
4	(4, 2)	-	(2 2 8)	-	-	(1 0 1 0 1 0 0 0 0 0 0 0)	-
5	(5, 1)	(1 7 4)	-	(1 4 7)	(1 1 0 0 0 0 0 0 1 0 0 0)	-	(1 1 0 0 0 1 0 0 0 0 0 0)
6	(5, 2)	(2 7 3)	-	(2 3 7)	(1 0 1 0 0 0 0 0 0 1 0 0)	-	(1 0 1 0 0 1 0 0 0 0 0 0) .
7	(6, 1)	(1 6 5)	-	(1 5 6)	(1 1 0 0 0 0 0 1 0 0 0 0)	-	(1 1 0 0 0 0 1 0 0 0 0 0)
8	(6, 2)	(2 6 4)	-	(2 4 6)	(1 0 1 0 0 0 0 0 1 0 0 0)	-	(1 0 1 0 0 0 1 0 0 0 0 0)
9	(6, 3)	-	(3 3 6)	-	-	(1 0 0 1 0 0 1 0 0 0 0 0)	-
10	(7, 2)	-	(2 5 5)	-	-	(1 0 1 0 0 0 0 1 0 0 0 0)	-
11	(7, 3)	(3 5 4)	-	(3 4 5)	(1 0 0 1 0 0 0 0 1 0 0 0)	-	(1 0 0 1 0 0 0 1 0 0 0 0)
12	(8, 4)	-	(4 4 4)	-	-	(1 0 0 0 1 0 0 0 1 0 0 0)	-

You will note that not every rhythm has its own evenness value. This is because rhythms can be both chiral and achiral: if a rhythm is chiral, then the rhythm with bits ordered in reverse is not identical to the original. So, chiral rhythms come in pairs, and since the distribution of bits only reverses order between rhythms in a chiral pair, evenness is the same for each. Achiral rhythms are equivalent to their own reverse ordering; they are palindromes, or more precisely, there exists a rotation that satisfies the sequence definition of a palindrome. Therefore, achiral rhythms are always found alone.

In the set of rhythms with length $k = 12$ and weight $n = 3$, there are 5 achiral rhythms and $2 \cdot 7$ chiral rhythms. Due to this chiral pairing situation, this means that while there are 19 distinct rhythms in this set, there are only 12 distinct evenness classes that these 19 rhythms can fall into.

Below is the same as above but for the rhythms of length $k = 12$, and $n = 9$:

Even#	*S*	$C_{12,9}$			*R*		
1	(8, 7, 6, 5, 4, 3, 2, 1)	-	(1 1 1 1 1 1 1 1 4)	-	-	(1 1 1 1 1 1 1 1 1 0 0 0)	-
2	(9, 7, 6, 5, 4, 3, 2, 1)	(1 1 1 1 1 1 1 2 3)	-	(3 2 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 0 1 0 0)	-	(1 0 0 1 0 1 1 1 1 1 1 1)
3	(9, 8, 6, 5, 4, 3, 2, 1)	(1 1 1 1 1 1 2 1 3)	-	(3 1 2 1 1 1 1 1 1)	(1 1 1 1 1 1 1 0 1 1 0 0)	-	(1 0 0 1 1 0 1 1 1 1 1 1)
4	(9, 8, 7, 5, 4, 3, 2, 1)	(1 1 1 1 1 2 1 1 3)	-	(3 1 1 2 1 1 1 1 1)	(1 1 1 1 1 1 0 1 1 1 0 0)	-	(1 0 0 1 1 1 0 1 1 1 1 1)
5	(9, 8, 7, 5, 4, 3, 2, 1)	(1 1 1 1 2 1 1 1 3)	-	(3 1 1 1 2 1 1 1 1)	(1 1 1 1 1 0 1 1 1 1 0 0)	-	(1 0 0 1 1 1 1 0 1 1 1 1)
6	(10, 8, 6, 5, 4, 3, 2, 1)	-	(1 1 1 1 1 1 2 2 2)	-	-	(1 1 1 1 1 1 1 0 1 0 1 0)	- .
7	(10, 8, 6, 5, 4, 3, 2, 1)	(1 1 1 1 1 2 1 2 2)	-	(2 2 1 2 1 1 1 1 1)	(1 1 1 1 1 1 0 1 1 0 1 0)	-	(1 0 1 0 1 1 0 1 1 1 1 1)
8	(10, 8, 7, 6, 4, 3, 2, 1)	(1 1 1 1 2 1 1 2 2)	-	(2 2 1 1 2 1 1 1 1)	(1 1 1 1 1 0 1 1 1 0 1 0)	-	(1 0 1 0 1 1 1 0 1 1 1 1)
9	(10, 8, 7, 6, 5, 3, 2, 1)	-	(1 1 1 2 1 1 1 2 2)	-	-	(1 1 1 1 0 1 1 1 1 0 1 0)	-
10	(10, 9, 7, 6, 4, 3, 2, 1)	-	(1 1 1 1 2 1 2 1 2)	-	-	(1 1 1 1 1 0 1 1 0 1 1 0)	-
11	(10, 9, 7, 6, 5, 3, 2, 1)	(1 1 1 2 1 1 2 1 2)	-	(2 1 2 1 1 2 1 1 1)	(1 1 1 1 0 1 1 1 0 1 1 0)	-	(1 0 1 1 0 1 1 1 0 1 1 1)
12	(10, 9, 8, 6, 5, 4, 2, 1)	-	(1 1 2 1 1 2 1 1 2)	-	-	(1 1 1 0 1 1 1 0 1 1 1 0)	-

You might notice that this evenness sorting approach does not place rhythms into the same evenness levels as their complements (switching 1s to 0s and vice versa). For example: (1 0 1 0 1 0 0 0 0 0 0 0) has evenness level 4, whereas (0 1 0 1 0 1 1 1 1 1 1 1) has evenness level 6. This is due to the musical application I made the algorithm for, which necessitated more even rhythms having minimal largest subsequences of 0s. To make rhythms and their complements align in evenness level, a slight alteration can be made to the algorithm: when $n > \frac{k}{2}$, define $S(C(R)) = \left(S_0(C(R)), ..., S_{n-2}(C(R))\right)$, and then sort with these sequences lexicographically.

In [12], Godfried Toussaint showed that maximally even rhythms are extremely common in music cross-culturally. While much of the literature on evenness on rhythmss has focused on maximal evenness, very little has been done to classify rhythms (or harmonies) by evenness level.

So, given this evenness property on rhythms, I am curious how to extend the evenness notion to tilings of rhythms. That is, when we tile rhythms of various evenness levels together, how do these evenness values affect the structure of the rhythm tiling? To this end, I believe a good place to start is by considering the optimized case of rhythm tilings involving maximally even rhythms.

Other than evenness measures or orderings, which both allow us to sort rhythms of equal length and weight, there also exists an extensive (and recent) literature on maximally even rhythms. Maximally even rhythms are also referred to as Euclidean rhythms. There are many definitions of Euclidean rhythms, here is the simplest:

Definition (Euclidean rhythm (Clough-Douthett [1])):

A rhythm R of length k and weight n is Euclidean, denoted $E_{k,\,n}$, if the rhythm generated by the Clough-Douthett algorithm is R. This algorithm is as follows:

$$C(E_{k,\,n}) = \left(\left\lfloor \frac{k(i+1)}{n} \right\rfloor - \left\lfloor \frac{ki}{n} \right\rfloor, i = 0..n-1\right).$$

Here is, in my opinion, the most structurally interesting construction of the Euclidean rhythm:

Definition (Euclidean rhythm (Demaine [14])):

Let concatenation be denoted by $\cdot$ and exponentiation. Given $k, n \in \mathbb{Z}^+$ such that $k \geq n$, let $r = k \bmod n$, and $C(E_{n,\,r}) = (x_0, x_1, ..., x_{r-1})$, then
If n divides k,

$$C(E_{k,\,n}) = \left(\frac{k}{n}\right)^n,$$

and if n does not divide k,

$$C(E_{k,\,n}) = \left(\left\lfloor \frac{k}{n} \right\rfloor^{x_0 - 1} \cdot \left\lceil \frac{k}{n} \right\rceil\right) \cdot \left(\left\lfloor \frac{k}{n} \right\rfloor^{x_1 - 1} \cdot \left\lceil \frac{k}{n} \right\rceil\right) \cdots \left(\left\lfloor \frac{k}{n} \right\rfloor^{x_{r-1} - 1} \cdot \left\lceil \frac{k}{n} \right\rceil\right).$$

The Demaine algorithm is interesting because it reveals the recursive nature of Euclidean rhythms, which mirrors the recursive structure of the Euclidean algorithm. Although interestingly, these rhythms were called Euclidean, not because of the Demaine algorithm, but because of Bjorklund's algorithm (see [9]), which constructs Euclidean rhythms using a totally different recursive concatenation scheme. In my research paper on Euclidean rhythms [17], I provided a proof that the objects resulting from Bjorklund's algorithm and Demaine's algorithm are equivalent, and I did this by showing the similarity of their respective concatenation schemes.

Euclidean rhythms have a lot of interesting properties in their own right as can be seen from these papers [10], [13], and [15]. Most relevant to our discussion here is that they are unique for a given k and n. More interesting work remains to be done on them!

I bring up Euclidean rhythms in the context of rhythm tilings because they enable us to define precisely an optimized case of rhythm tiling. That is, when all rhythms in the tiling are maximally even. I am calling this a Euclidean rhythm tiling:

Definition (Euclidean Rhythm Tiling):

A Euclidean rhythm tiling is a rhythm tiling in which each of its rhythms is Euclidean. Symbolically:

$$\begin{bmatrix} E_{k,\,n_0} \\ \vdots \\ E_{k,\,n_2} \end{bmatrix} \xrightarrow{Tiling} [\textit{rhythm tiling}].$$

For example, both the following are different Euclidean rhythm tilings with $m = 4$, $k = 9$. It may also be worth noting that the order of Euclidean rhythms in the tilings (that is, the row order. See above note on row ordering) is also the same, which implies that Euclidean rhythm tilings are not necessarily unique.

$$A = ERT_{4,\,9,\,(4,\,2,\,2,\,1)} = \begin{bmatrix} ER_{9,\,4} \\ ER_{9,\,2} \\ ER_{9,\,2} \\ ER_{9,\,1} \end{bmatrix} \xrightarrow{Tiling} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \end{bmatrix},$$

$$B = ERT'_{4,\,9,\,(4,\,2,\,2,\,1)} = \begin{bmatrix} ER_{9,\,4} \\ ER_{9,\,2} \\ ER_{9,\,2} \\ ER_{9,\,1} \end{bmatrix} \xrightarrow{Tiling} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

Goal and Initial Questions:

I am curious to know when Euclidean rhythm tilings exist for given m, k, and weight necklace, and how many there are when they do. Ultimately, I would like to determine the relationship between rhythm tiling existence (as well as number) and the level of evenness of the component rhythm tiles that make up the tiling. For instance, do more or less tilings exist when rhythm tiles have similar evenness? Is existence guaranteed for Euclidean rhythm tilings? Can we construct an infinite family of Euclidean rhythm tilings? Can we construct a family of rhythm tilings constituting rhythm tiles with different evenness values? If so, is there a bound on the evenness value disparity between these rhythm tiles? What is it?

I am sure there are many other interesting questions to ask here!

Some First Steps:

1. Review literature on similar objects and their tiling properties

2. Investigate questions of existence and enumeration regarding Euclidean rhythm tilings

3. Review similar objects that seem to have high evenness (For example: Costas arrays appear to have high evenness. Is there a vector distinctness related reason for this?).

4. Review literature on evenness, and evenness-like measures on related combinatorial objects

5. Formulate conjectures and prove/falsify them!

Papers to help get started:

[13] Interlocking and Euclidean Rhythms by F. Gomez-Martin, P. Taslakian, and G. Toussaint:

This paper appears to have found some general results on a particular family of Euclidean rhythm tilings when the number of 1s in each Euclidean rhythm differs by at most 1.

[16] An Algebra for Periodic Rhythms and Scales by E. Amiot and W. Sethares:

Amiot and Sethares explore linear algebra properties of circulent matrices created from the rotations of a given rhythm. They have a brief section on tilings that seems to relate these circulent matrices to tiled rhythm canons.

[7] Tiling Problems in Music Composition: Theory and Implementation by M. Andreatta, C. Agon, and E. Amiot:

Contrary to the general title of the paper, this paper only discusses rhythm canons and how the OpenMusic software is useful at working with them. Theoretically, this paper discusses how rhythm canons can be viewed as products of polynomials over $\mathbb{Z}_2$, as well as a factoring of cyclic group $\mathbb{Z}_n$ into a direct sum of subsets of said group, each of which correspond to the indices of 1s in two special rhythms which then generate the canon.

[6] Enumeration of Non-Isomorphic Canons by H. Fripertinger:

This paper appears to be highly cited in the rhythm canon literature.

[11] Webpage of a transcript of a lecture titled "Perfect Rhythmic Tilings" by T. Johnson*:

Johnson discusses a special case of rhythm tiling, called perfect rhythm tilings, whereby for each rhythm tile, between each (save one) of the 1s is the same number of 0s. The perfect rhythm tiling mirrors the notion of a perfect tiling used in other contexts.

*There is a 2011 paper with this title by Jean-Paul Davalan that looks quite relevant, however I am having trouble accessing it since I am no longer affiliated with an institution.

Reference List:

[1] J. Clough and J. Douthett, "Maximally even sets," *J. Music Theory*, vol. 35, pp. 93-173, 1991.

[2] D. T. Vuza, "Supplementary sets and regular complementary unending canons (part one)," *Perspectives of New Music*, vol. 29, no. 2, pp. 22-49, 1991.

[3] D. T. Vuza, "Supplementary sets and regular complementary unending canons (part two)," *Perspectives of New Music*, vol. 30, no. 1, pp. 184–207, 1992.

[4] D. T. Vuza, "Supplementary sets and regular complementary unending canons (part three)," *Perspectives of New Music*, vol. 30, no. 2, pp. 102–125, 1992.

[5] D. T. Vuza, "Supplementary sets and regular complementary unending canons (part four)," *Perspectives of New Music*, vol. 31, no. 1, pp. 270–305, 1993.

[6] H. Fripertinger, "Enumeration of non-isomorphic canons," *Tatra Mountains Mathematical Publications*, vol. 23, pp. 47-57, 2001.

[7] M. Andreatta, C. Agon, E. Amiot, "Tiling problems in music composition: Theory and Implementation," 2002.

[8] E. Bjorklund, "A metric for measuring the evenness of timing system rep-rate patterns," *SNS ASD Technical Note SNS-NOTE-CNTRL-100*, Los Alamos National Laboratory, Los Alamos, USA, 2003.

[9] E. Bjorklund, "The theory of rep-rate pattern generation in the SNS timing system," Los Alamos, NM: Los Alamos National Laboratory, 2003.

[10] J. Ellis, F. Ruskey, J. Sawada and J. Simpson, "Euclidean strings," *Theoretical Computer Science*, no. 301, pp. 321-340, 2003.

[11] T. Johnson, "Perfect rhythmic tilings," *Lecture delivered at MaMuX meeting during IRCAM*, 2004.
link: http://recherche.ircam.fr/equipes/repmus/mamux/documents/Perfectrhythmictilings.html

[12] G. Toussaint, "The Euclidean algorithm generates traditional musical rhythms," 2005.
link: http://cgm.cs.mcgill.ca/~godfried/publications/banff.pdf

[13] F. Gomez-Martin, P. Taslakian and G.Toussaint, "Interlocking and Euclidean rhythms," *Journal of Mathematics and Music*, vol. 3, no. 1, pp. 15-30, 2009.

[14] E. D. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G. T. Toussaint, T. Winograd and D. R. Wood, "The distance geometry of music," *Computational Geometry: Theory and Applications*, no. 42, pp. 429-454, 2009.

[15] F. Gomez-Martin, P. Taslakian and G. Toussaint, "Structural properties of Euclidean rhythms," J*ournal of Mathematics and Music*, vol. 3, no. 1, pp. 1-14, 2009.

[16] E. Amiot, W. Sethares, "An algebra for periodic rhythms and scales," *Journal of Mathematics and Music*, vol. 5, 2009.

[17] T. Gaede, "Euclidean rhythms: an investigation into the structure of maximal and general evenness of rhythms," 2018.
link: https://taogaede.com/wp-content/uploads/2019/11/Final-Report-With-Revisions-Tao-Gaede505.pdf