Euclidean Rhythms: An Investigation into the Structure of Maximal and General Evenness of Rhythms

By Tao Gaede

For Dr. Wayne Broughton

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Introduction

Consider a binary necklace of length *k* with *n* 1s and *k* - *n* 0s. How does one organize these bits such that all 1s are maximally evenly distributed amongst the 0s? The solution may seem simple at first glance, however maximal evenness in an exact sense turns out to be a difficult notion to define, thereby making the above problem nontrivial to resolve in all cases. When n divides *k*, intuition reliably leads to the correct solution to the original question above as it will dictate that there must be $k/n - 1$ 0s between each 1. However, the solution becomes nontrivial when k and n are relatively prime since there must be a variable number of 0s between each 1 to satisfy the initial length and bit frequency conditions. For example, when *k* is 18 and *n* is 6, the maximally even necklace is (1,0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,0,0), 18/6 – 1 = 2 0s between each 1. However, suppose *k* is 12 and *n* is 7, then the maximally even necklace is (1,0,1,0,1,1,0,1,0,1,0,1). Notice that there are no 0s between 1s in two cases and in five cases there is one 0 between 1s. While the attentive reader may note that 5 = 12 mod 7 and 2 = 7 – 5 and rightly hypothesize formulas for the sizes of the two distinct 0 subsequences between the 1s, it is not obvious where exactly these 0 subsequences should and should not be placed. Upon studying these examples, it should be apparent that a precise characterization of maximal evenness on binary necklaces ought to be a deep one. The development of the mathematical investigation of the property of maximal evenness on two colour necklaces, here called Euclidean rhythms, is the primary topic investigated in the first two sections of this paper.

Euclidean rhythms (ERs), when considered as musical rhythms are prevalent rhythms in music generally, and their mathematical study began quite recently in a 2005 paper titled "The Euclidean Algorithm Generates Traditional Music Rhythms" by Godfried Toussaint. Toussaint was the first author to discuss the prevalence of Euclidean rhythms in music, and significant mathematical investigation of ERs began shortly after Toussaint's first paper on the subject. It was discovered from this investigation that several other disparately defined objects from the combinatorics of words to the mathematical music theory literature all appear to be equivalent to ERs. Some of this equivalency work remains to be done however, and in section 2 of this paper are results to this end.

Overview

Section 1 of this paper is a review of the mathematical characterization maximally evenness on necklaces, here called rhythms (see definition 1.1.1.1). The aim of this exposition is to summarize key results around maximally even rhythms and to identify results that may be needed but are missing from the literature. Section 2 contains novel mathematical results pertaining to the study of maximally even rhythms. Section 3 is an introduction to a collection of compositions that are equivalent to rhythms.

Various definitions and properties around these compositions will be proven. Section 4 is a brief discussion on general evenness on rhythms using an example of a particular evenness measure. Section 5 contains a mathematical construction that classifies the compositions defined in section 3 according to a given evenness measure. Such a classification is useful because these compositions can represent musical rhythms as well as chords and scales.

1. Review

1.1. Introduction

1.1.0. Overview

This section constitutes a discussion on the development of a class of objects called Euclidean rhythms (ERs). There appears to be two different classes of objects both called ERs, and it is assumed that the two classes are equivalent, but this equivalence has not yet been proven. So, in addition to a presentation of the landscape and development of the two distinct classes, the primary argument in this section is to establish the need for a proof of their equivalence.

Both ER classes are special cases of objects called rhythms. In this context, rhythm has a complex definition because it is an object with three equivalent forms that are each superficially different from one another. To distinguish between the two ER classes, the terms class one and class two will be used, and a brief outline of the class one and two constructions follows:

The most important property of class one ERs is maximal evenness as defined below in Definition 1.1.3.1. A rhythm is class one Euclidean if and only if it is maximally even. Since there are three equivalent forms of a rhythm, there are similarly three equivalent definitions of maximal evenness corresponding to each rhythm form. To minimize confusion, all these definitions and equivalencies will be provided at an early stage in this discussion.

Class two ERs are defined simply as rhythms resulting from an algorithm called Bjorklund's algorithm, which is an algorithm that is intuited from examples to produce rhythms of high evenness[1]. Nowhere is it shown, however that the rhythms from Bjorklund's algorithm have a precisely defined characteristic identical to the maximal evenness of class one ERs. So, contrary to class one ERs, class two ERs are not defined by their structural characteristics, but they are instead defined by how they are constructed. In other words, class two ERs have not been shown to satisfy Definition 1.1.3.1, so it is not known whether they are maximally even.

[1] A detailed description of Bjorklund's algorithm will be provided later.

This distinction between class one and two ERs is not clearly identified in the literature. It is assumed that both types of ERs are equivalent while no proof of this equivalence exists, and some papers have even produced results based on this assumption. Thus, a key goal of this section is to indicate the importance of determining via proof whether this equivalence assumption is valid.

To begin, since many terms are used in the literature to describe the equivalent forms of rhythm and maximal evenness, a more organized construction of the two ER classes will be made here for later reference. First, the definition of a rhythm will be given, along with its three equivalent forms. The three equivalent definitions of maximal evenness will follow, and a formal presentation of Bjorklund's algorithm will be presented. This introduction will conclude with a description of precisely how class one and two ERs are characterized.

1.1.1. Rhythms

Given some number of time units at which an event occurs at only some of these time units, a rhythm is an object that describes exactly at which of the time units events occur, relative to other event occurrences. There are three equivalent ways of conveying this information, and so it can be said that rhythms have three equivalent forms. First, a definition of the generic rhythm will be given, followed by formal definitions and examples of the equivalent forms. To emphasize equivalence, the same parameters will be used for the examples.

Definition 1.1.1.1. (Rhythm):

Let $k, n \in \mathbb{Z}$ where $0 \leq n \leq k$, where k denotes the number of time units, and n denotes the number of time units at which an event occurs. A rhythm R is an object that describes the relative positioning of the k time units at which n events occur[2].

The information of R can be expressed in the following three ways:

Binary Necklace Rhythm:

R is a binary necklace of length k and weight n. The 0s and 1s describe time units at which 1s indicate an event occurrence, and 0s indicate no event occurrence. For example, let k = 18 and n = 13, then we could have the following two binary necklaces:

a. (1,0,1,1,0,1,1,1,0,1,1,1,0,1,1,0,1,1)
b. (1,1,1,1,0,0,1,1,0,0,1,1,1,0,1,1,1,1)

[2] This definition of rhythm should not be confused with the musical definition. Definition 1.1.1.1. is more general than the musical notion of rhythm, and the nominal similarity is unfortunate. In fact, rhythms will be used by some authors in the literature to represent non-rhythm musical objects like chords and scales.

Integer Necklace Rhythm:

R is an integer necklace of length *n* and weight *k* where each element of the necklace describes the number of time units between consecutive event occurrences. Example:

a. (2,1,2,1,1,2,1,1,2,1,2,1,1)
b. (1,1,1,3,1,3,1,1,2,1,1,1,1)

Subset Class Rhythm:

R is a collection of sets $A \subseteq \{0, 1, ..., k-1\}_n$ such that each *A* produces the same necklace of differences between consecutive elements. The elements of each set in the collection describe the time unit between 0 and *k* - 1 at which an event occurs. For example, using the same parameters as above, we get:

a. [{0,2,3,5,6,7,9,10,11,12,14,15,17}]
b. [{0,1,2,3,6,7,10,11,12,14,15,16,17}]

Where the square brackets indicate the class of subsets of $\{0, 1, ..., k-1\}_n$ that produce the same integer necklace of differences between consecutive elements as the representative set contained within the square brackets.

<u>1.1.2 Equivalencies Between Rhythm Forms</u>

Between the Integer and Binary Rhythms:

Given a rhythm *R* in binary form denoted $R_B = (b_0, b_1, ..., b_{k-1})$, the equivalent integer necklace representation, denoted R_I, is given by the necklace of differences between the indices of consecutive 1s in R_B. The following two operations, first described in [1], will be used repeatedly throughout this paper and they succinctly describe the equivalency between binary and integer rhythms.

<u>Definition 1.1.2.1. (Delta and Delta Inverse):</u>

$\delta(R_I)$ applies to each element of R_I, the morphism $i \to 0^{i-1}1$, where *i* is an integer.

$\delta^{-1}(R_B)$ applies to each subsequence of R_B immediately following a one, the morphism $0^{i-1}1 \to i$.

Therefore $\delta(R_I) = R_B$, and $\delta^{-1}(R_B) = R_I$.

Between the Subset and Binary Rhythms:

Given the binary rhythm $R_B = (b_0, b_1, ..., b_{k-1})$, for every *j* rotation of R_B , define the set A_j to be the set of indices of the *j* rotation of R_B containing 1s. The subset rhythm R_S is defined as the class of A_j s for all $j \in [0, k-1]$.

Conversely, choose some set A in R_S , then define the binary necklace $R_B = (b_0, b_1, ..., b_{k-1})$ such that for each $a_j \in A, b_j = 1$ and assign a 0 to every other element.

Between the Subset and Integer Rhythms:

Given the subset rhythm R_s , by definition, each set in R_s corresponds to the same integer necklace of differences between consecutive elements. Define the integer rhythm R_I to be this unique integer necklace. Conversely, given R_I , the subset rhythm R_s is the class of subsets of $\{0, 1, ..., k-1\}_n$ such that for each of these subsets, the integer necklace of differences between consecutive elements is R_I .

With the three rhythm forms established as equivalent, a rhythm *R* with parameters *k* and *n* defined in Definition 1.1.1.1. will now be referred to in terms of its integer rhythm form where *k* is weight and *n* is length. So, for the remainder of this paper, if the form of a given rhythm is not explicitly stated, its *k* value will be called weight and its *n* value will be called length. By abuse of notation, since the rhythm *R* is equivalent to R_I, R_B, and R_S, it will be notated that $R = R_I = R_B = R_S$.

1.1.3 Maximal Evenness in Class One ERs

The distinctive property that characterizes class one ERs, maximal evenness, will now be defined for each rhythm form. In the literature, maximal evenness is formally defined for only the subset rhythm form, however since rhythms have three equivalent forms, I will present for each of these rhythm forms an equivalent definition of maximal evenness.

Maximal Evenness:

Definition 1.1.3.1. (Rhythm Maximal Evenness):

Let *R* be a rhythm with weight *k* and length *n*. Then *R* is said to be maximally even if and only if it is maximally even in any of its forms:

a. $R_I = \left(I_0, I_1, ..., I_{n-1}\right)$ is maximally even if and only if

$\forall\ i \in [0, n-1], \forall\ l \in [1, n], \sum_{j=i}^{i+l-1} I_j \in \left\{ \left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil \right\}$.

b. $R_B = \left(B_0, B_1, ..., B_{k-1}\right)$ is maximally even if and only if $\forall\ i, l \in [0, k-1], \forall\ i < q < l$ where $B_i = B_l = 1$ and $B_q = 0$, it holds that $\left| B_{i+l} - B_i \right| \in \left\{ \left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil \right\}$.

c. R_S is maximally even if and only if $\forall\ S \in R_S, \forall\ s \in S, \forall\ i \in [0, n-1], \forall\ l \in [1, n]$, it holds that $\left| a_{i+l} - a_i \right| \in \left\{ \left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil \right\}$.

The equivalencies of the maximal evenness definitions for each form follow directly from the equivalencies of the forms described above.

A class one ER is a rhythm that is maximally even, and it is unique for every $0 < n \leq k$[3].

1.1.4 Class Two ERs and Bjorklund's Algorithm:

Class two ERs are defined as the binary rhythms derived from Bjorklund's algorithm in the following way: Bjorklund's algorithm produces binary sequences of length *k* and weight *n*. By taking the rotationally invariant form of these sequences, binary necklaces called Bjorklund necklaces are produced. Class two ERs are defined as Bjorklund necklaces, denoted $B_{k,n}$.

To reiterate an earlier note, even though class two ERs are defined as binary necklaces with length *k* and weight *n*, they are still rhythms and therefore have three equivalent forms. So, class two ERs are said to have weight *k* and length *n*.

Bjorklund's Algorithm:

Bjorklund's Algorithm was originally presented in [2] but the presentation in [3] is more succinct. What follows is Bjorklund's algorithm based on [3]'s presentation:

Bjorklund's algorithm is a 2-step algorithm, the first step initializes, and the second repeats until completion.

Given $k, n \in \mathbb{Z}^+$ where $k \geq n$,

First Step:

[3] Existence and uniqueness of class one ERs is proven independently by all three papers that produce salient results about them. Explicitly, it is shown by [4] [5], and implicitly, it is shown by [1].

1. Construct the string $1^n 0^{k-n}$, where juxtaposition is concatenation and superscript denotes number of juxtapositions.
2. If $n \leq k - n$ then define $A = 1^n, B = 0^{k-n}$, and $a = n, b = k - n.$ Otherwise define $A = 0^{k-n}, B = 1^n$, and $a = k - n, b = n$.
3. Beginning from the rightmost bit, remove $\left\lfloor \frac{k}{n} \right\rfloor$ strings of a consecutive bits from B, and place them underneath the string A forming a $\left(\left\lfloor \frac{k}{n} \right\rfloor + 1\right) \times a$ matrix.
4. Redefine A as this matrix and B as the length $(b \bmod a)$ string of bits and redefine $b = (b \bmod a)$.

Second Step:

1. Beginning from the rightmost column of B, remove $\left\lfloor \frac{a}{b} \right\rfloor$ columns of B and place them in a row below the leftmost column of A.
2. Do the same for the first $\left\lfloor \frac{a}{b} \right\rfloor$ rightmost columns of A, but place them below the former columns of B. What results is a new matrix.
3. Redefine A as the first b columns of this matrix, and B as the remaining $(a \mathbf{\bmod} b)$ columns of the matrix. Then redefine $b_{new} = (a_{old} \bmod b_{old})$, and $a_{new} = b_{old}$.
4. If B has more than one column, then repeat Step 2.

If B is empty or has only one column, then the algorithm stops and outputs from the leftmost column of A, the columns of A concatenated as strings with the column of B. That is, the Bjorklund necklace is defined by this string such that $B_{k,n} = (A_1, A_2, \ldots, A_p, B)$ where A_i are the columns of A, and p is the number of columns of A.

Notice the significant difference in the characterizations of class one and class two Euclidean rhythms. It is not obvious from the Bjorklund algorithm whether Bjorklund necklaces are maximally even. Regarding notation, class one ERs will be denoted $ER_{k,n}$ while class two ERs will be denoted $B_{k,n}$, where k is weight and n is length.

1.2. Literature Review

The literature review will proceed chronologically in two parts. First the development of class one ERs will be discussed, and then the development of class two ERs will follow.

1.2.1.0 Class One ERs:

The literature on class one ERs is complex because it consists of two disparately developed objects, which both turn out to characterize maximally even rhythms. The first such object comes from the mathematical music theory literature and is initially defined in subset class form by [4], and then generalized by [5] to be equivalent to all three rhythm forms. The second object studied in [1] is interestingly neither studied for its maximal evenness, and nor its application to music, but it nonetheless characterizes both integer and binary rhythms for different values of *k* and *n*.

1.2.1.1 "*Maximally Even Sets"* - Clough & Douthett [4]

The formal study of maximal evenness appears to begin with a mathematical music theory paper titled "*Maximally Even Sets*" by Clough and Douthett. This paper is complicated in that many terms are defined and given music theory inspired names even when standard mathematical ones exist. For example, the terms "diatonic length" and "chromatic length" refer to the length and weight of a given integer subsequence, respectively. While the terminology is a bit obstructive to the non-musical reader, the paper is well worth the attention because in it, several salient mathematical results are proven. The primary objects of study in this paper are subset class ERs with weight *k* and length *n*, denoted in this paper as ME(k,n). The result from [4] most cited in the later more mathematical literature is the Clough-Douthett algorithm, which is used to generate the particular sets in ME(k,n) called maximally even (ME) sets, denoted $ME_{k,n}$. The algorithm is as follows:

Given $k, n \in \mathbb{Z}^+ \ s.t. \ k \geq n$, then

$ME_{k,n} = \left(\left\lfloor \frac{ik}{n} \right\rfloor + j : i \in [0, n-1] \right), j \in \{0, 1, ..., k-1\}$, where *j* identifies the specific set in ME(k,n) that is generated.

The following variant of this algorithm is also used to generate the integer necklace ER equivalent to ME(k,n):

$$ME(k,n) = ER_{k,n} = \left(\left\lfloor \frac{(i+1)\cdot k}{n} \right\rfloor - \left\lfloor \frac{i \cdot k}{n} \right\rfloor : i \in [0, n-1] \right).$$

For example, when *k* = 13 and *n* = 7 it is the case that,

$$ME(13,7) = \left(\left\lfloor \frac{1\cdot 13}{7} \right\rfloor - \left\lfloor \frac{0\cdot 13}{7} \right\rfloor, \left\lfloor \frac{2\cdot 13}{7} \right\rfloor - \left\lfloor \frac{1\cdot 13}{7} \right\rfloor, \left\lfloor \frac{3\cdot 13}{7} \right\rfloor - \left\lfloor \frac{2\cdot 13}{7} \right\rfloor, \right.$$
$$\left\lfloor \frac{4\cdot 13}{7} \right\rfloor - \left\lfloor \frac{3\cdot 13}{7} \right\rfloor, \left\lfloor \frac{5\cdot 13}{7} \right\rfloor - \left\lfloor \frac{4\cdot 13}{7} \right\rfloor, \left\lfloor \frac{6\cdot 13}{7} \right\rfloor - \left\lfloor \frac{5\cdot 13}{7} \right\rfloor, \left\lfloor \frac{7\cdot 13}{7} \right\rfloor$$
$$\left. - \left\lfloor \frac{6\cdot 13}{7} \right\rfloor \right)$$
$$= (1-0, 3-1, 5-3, 7-5, 9-7, 11-9, 13-11) = (1, 2, 2, 2, 2, 2, 2).$$

Other than this algorithm, which is proved to produce class one ERs, many other results in [4] are unfortunately not much developed in the later literature even though they pertain to properties of class one ERs. In fact, a paper to be discussed later [5], has much of its mathematical investigation devoted to reproving that the Clough-Douthett algorithm produces class one ERs and that they are unique. While [4] is cited in most of the later literature on both ER classes, the only result typically cited is the Clough-Douthett algorithm. It may be the case that there is a musical jargon barrier blocking the permeation of mathematical results from the mathematical music theory literature.

In equivalent terms, [4] defines an ME set M as a length n set of integers from the set $\{0, 1, \ldots, k - 1\}$ such that for every integer l, the differences between each integer in M and the integer l steps to the right modulo n are equal to at most two values for each $0 < l < n - 1$. Later in the beginning section of their paper, it is shown in a lemma that these difference values for a given l are $\left\{\left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil\right\}$. Therefore every set in ME(k, n) is maximally even in this sense and so [4] defines maximal evenness equivalently to Definition 1.1.3.1.b. [4] appears to be the first paper to perform an in-depth study on maximal evenness on subset class rhythms[4]. The most important result, as it pertains to this study, is that for any k and n such that $0 < n \leq k$, ME(k,n) exists and is unique. Since ME(k,n) = $ER_{k,n}$, this result is equivalent to saying that $ER_{k,n}$ exists and is unique for $0 < n \leq k$.

Following their existence and uniqueness result, [4] proves that there are k / (k, n) distinct ME sets for any k and n where $k \geq n$; that is, | ME(k,n) | = k / (k,n). For example: If k = 8, and n = 6, then we have (8,6) = 2 and ME(8,6) = (2, 1, 1, 2, 1, 1), where {0, 2, 3, 4, 6, 7}, {1, 3, 4, 5, 7, 0}, {2, 4, 5, 6, 0, 1}, and {3, 5, 6, 7, 1, 2} are the only distinct ME sets that exist in ME(8, 6)[5]. A similar result is that there are n / (k,n) ME sets in ME(k,n) that contain a particular element of in $\{0, 1, \ldots, k - 1\}$. For example: consider the same parameters as before, then for element 0, we get {0, 2, 3, 4, 6, 7}, {0, 1, 2, 4, 5, 6}, and {0, 1, 3, 4, 5, 7}, which are the only distinct ME sets of ME(k,n) containing 0. This is the first result in the ER literature to reveal a connection between $ER_{k,\,n}$ and k / (k,n) and n / (k,n) when (k,n) > 1. Another such result, will be proven in Lemma 2.1.3 of this paper. To anticipate this result, the reader is encouraged to calculate the weight and length of the integer necklace rhythm (2, 1, 1) and relate it to the integer necklace form of $ER_{8,6}$ above.

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[4] The first formulation of Definition 1.1.3.1.b appears to be made in [13], however [4] mentions that only the case when k and n are relatively prime is considered.

[5] Note that these sets make up the blocks of an (8,4,3,6,2) balanced incomplete block design, which suggests a relationship between maximal evenness and design theory when (k,n) > 1.

In section 2 of [4], a subtle result suggesting that class one ERs for certain k and n values have a higher level or more constrained maximal evenness character than other values is given. This result follows: if $n \leq k / 2$, then for all distinct integers l_1 and l_2 and every set M in ME(k,n), the differences between a given element in M and l_1 and l_2 integers to the right modulo n are never equal. Since these difference values overlap when $k / 2 < n \leq k$, maximal evenness seems to convey slightly different information for different k and n values.[6]

Lemma 2.3 shows the inefficiency of using subset class form to describe ERs, and its presentation in this form will be confined to a footnote.[7] Using integer necklace form, Lemma 2.3 becomes simply: if $n \mid k$, then $ER_{k,n}$ is n repetitions of k / n. This lemma presents the trivial case for class one ERs, whereby $ER_{k,n}$ contains only one distinct value as opposed to the usual two in integer necklace form. In their third section, [4] defines the complement of an ME set given by {0,1,…,k-1}\M, and they show that the complement of an ME set is also ME. This shows a correspondence between the sets ME(k, n) and ME(k, k-n), which equivalently defines a correspondence between $ER_{k,n}$ and $ER_{k,k-n}$. This correspondence is significant and will be revisited in a more general context applying to all rhythms in definition 3.1.7 of this paper. An algorithm for calculating ME(lk,n) for all $1 \leq l \leq n - 1$ when ME(k,n) is known is presented as well, but while interesting enough to be explained in a footnote, it is not relevant to this exposition.[8]

[4] was interested in maximal evenness insofar as the property applied to integer sets corresponding to music chords and scales. They note that chords and scales that are maximally even appear frequently in music. Clough and Douthett appear to be mainly interested in determining how the maximal evenness property might explain the high prevalence of these chords and scales in music.[9] As

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[6] For ME(12,5), we get difference values for l in [0,4]: 2-3, 4-5, 7-8, 9-10, and 12, the differences for different l values do not overlap. Whereas for ME(12,7), we get for l in [0,6]: 1-2, 3-4, 5-6, 6-7, 8-9, 10-11, and 12. Notice that for ME(12,7), 6 is a difference for $l = 2$ and $l = 3$.

[7] The lemma is as follows: if $n \mid k$, then there are no two distinct integers l_1 and l_2 such that for some M in ME(k,n) the differences between some integer in M and l_1 and l_2 integers to the right modulo n are not equal. In other words, for every integer l the difference between any integer in any set M in ME(k,n) and the integer l elements to the right modulo n is identical for every l, namely lk / n.

[8] While originally described in terms of subset classes, the algorithm has an equivalent integer necklace form:

1. Consider the integer necklace $ER_{k,n}$. Beginning from each entry of $ER_{k,n}$, sum l rightward entries and define a new integer necklace N assigning in order these sums as its values.
2. Beginning at an arbitrary element of N, construct a new integer necklace by assigning to it n rightward l multiple entries of N. This constructed integer necklace is $ER_{lk,n}$

For example, let $k = 12$, and $n = 7$, then $ER_{12,7}$ is (2,2,1,2,2,2,1). For $l = 2$, Step 1 of the calculation produces N = (4,3,3,4,4,3,3), and by skipping every 2nd element of N, step 2 gives us $ER_{24,7}$ = (4,3,4,3,3,4,3). Similarly, for $l = 4$, N = (7,7,7,7,7,7,6) giving us $ER_{48,7}$ = (7,7,7,7,7,6,7).

[9] To exemplify the motivation of [4], $ER_{12,7}$ = (2,2,1,2,2,2,1) can represent the diatonic scale where the integers are intervals between semitones (adjacent keys on the piano). The diatonic scale consists of 7 notes separated by the

expected from the paper's title "*Maximally Even Sets*", they are narrowly concerned with the notion of maximal evenness, and so the notion of an evenness measure applied generally to all chords and scales is left unexplored, however this will be a focus of section 4 of this paper. Aside from the next paper, the remaining papers are interested in maximal evenness in so far as it is applied to musical rhythms.

1.2.1.2 "*Euclidean Strings*" - Ellis et al. [1]

The next mathematically rigorous investigation of class one ERs comes from the combinatorics literature with a paper introducing an object called Euclidean strings [1]. Euclidean strings are not necklaces, however if an integer necklace is defined as the object with a Euclidean string as a particular rotation, then this necklace turns out to be $ER_{k,n}$ in integer necklace form. An equivalent form of this fact is proven in [1], which will be described shortly. Euclidean strings are defined as length n and weight k integer strings, denoted $E_{k,n}$, where $E_{k,n} = p = (p_0, p_1, ..., p_{n-1})$ has the property that there exists a d such that $\tau(p) = (p_0 + 1, p_1, ..., p_{n-1} - 1)$ is a d right rotation of p, denoted $\rho^d(p) = (p_{n-d}, p_{n-d+1}, ..., p_0, ..., p_{n-d-1})$, that is $\tau(p) = \rho^d(p)$. Ellis et al. call d the displacement of $E_{k,n}$ and they show that $E_{k,n}$ exists and is unique if and only if k and n are relatively prime.

Another Euclidean string parameter c, called the cost of $E_{k,n}$, is defined as $c = \sum_{i=0}^{d-1} E_{k,n}$. d and c are shown to be the multiplicative inverse of k modulo n and $(k - n)$ modulo n, respectively. Euclidean strings are characterized as strings of (k mod n) $\left\lceil \frac{k}{n} \right\rceil$ elements and (n – (k mod n)) $\left\lfloor \frac{k}{n} \right\rfloor$ elements where the ceiling elements have indices $\{(n-1+jd) \bmod n : j \in [0, r-1]\}$ and floor indices $\{jd \bmod n : j \in [0, n-r-1]\}$.

Notice that the notion of maximal evenness is nowhere explicit in the definition or characterization of the Euclidean string. The maximal evenness result in this paper comes unbeknownst to the authors from their final theorem, which states that a Euclidean string of weight k and length n such that $0 < k < n$ is something called the "rational mechanical word" for the rational number $\frac{k}{n}$, denoted $t_{k,n}$, where $t_{k,n} = \left(\left\lfloor \frac{(i+1) \cdot k}{n} \right\rfloor - \left\lfloor \frac{i \cdot k}{n} \right\rfloor : i \in [0, n-1] \right)$. Notice that this definition is identical to the integer necklace form of the Clough-Douthett algorithm. Since rhythms are not defined when $k < n$, this result implies that all Euclidean strings $0 < n \leq k$ define integer necklace rhythms that are maximally even. But briefly consider the $k < n$ case for the weight and length of Euclidean strings, and notice that these Euclidean strings are binary sequences of length n with weight $r = k$ mod n. Binary Euclidean

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intervals described in $ER_{12,7}$, it is the most familiar scale in all of western music, and it is maximally even. [4] is the only paper in this exposition that uses ERs to represent chords and scale strings still have their ceiling and floor elements arranged in the same order as any other Euclidean string, which is, by the above theorem in [1], maximally even. So, since $n > r$, it follows that maximally even binary necklace rhythms can be defined from binary Euclidean strings. Therefore, all Euclidean strings correspond to class one ERs.

Ellis et al. define a collection of operations on Euclidean strings that preserve the Euclidean string characterization up to rotation. These operations include those of Definition 1.1.2.1 as well as two others defined here and will be referred to later:

Definition 1.2.1.2.1 (Euclidean String Operations):

Let a be a Euclidean string of weight k and length n. Then define

a. $R(a)$ is the reverse or mirror image of a, that is $R((a_0, a_1, \ldots, a_{n-1})) = (a_{n-1}, a_{n-2}, \ldots, a_0)$

b. $Inc^i(a)$ increments each element of a by i, where i is an integer.

[1] shows that binary Euclidean strings are concatenations of two smaller binary Euclidean strings, and that this concatenation is related to sequences called Farey sequences. When $k < n$, a connection between Farey sequences and the k, n, c, and d parameters of $E_{k,n}$ is mentioned, but not precisely stated in [1], so this connection will be stated here and developed in footnote 11. The Farey sequence of order n is the sequence of reduced fractions with denominators less than or equal to n ordered from lowest to highest value.[10] Suppose k and n are relatively prime with $k < n$, then $\frac{k}{n}$ is an element of the Farey sequence of order n. [1] presents a result equivalent to showing that $\frac{c}{d}$, and $\frac{k-c}{n-d}$ are always neighbours of $\frac{k}{n}$ in the Farey sequence of order n. It is then shown that $E_{k,n}$ is the concatenation of $E_{c,d}$ and $E_{k-c,n-d}$, evincing a connection between Euclidean string concatenation and adjacent elements in Farey sequences.[11]

[10] Fun fact: since Euclidean strings exist and are unique when k and n are relatively prime, the number of binary Euclidean strings with length less than or equal to n is the length of the Farey sequence of order n

$|F_n| = 1 + \sum_{i=1}^{n} \phi(i)$, where the summand is Euler's totient function.

[11] This concatenation scheme can be applied to all Euclidean strings. For example: Let k = 19 and n = 34. Then $E_{19,34} = (0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1)$. The neighbours of 19/34 in the Farey sequence of order 34 are 5/9 and 14/25.

$E_{5,9} = (0, 1, 0, 1, 0, 1, 0, 1, 1)$,

$E_{14,25} = (0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1)$.

The existence of a duality on Euclidean strings is proven using the composition $\delta\left(R\left(E_{k,n}\right)\right) = E_{n,k}$ where weight and length are switched. This duality result implies the weaker claim that δ preserves rotational equivalence to a Euclidean string. This rotational equivalence fact will be used in Definition 2.2.5 of this paper to generalize a class one ER producing algorithm (see Definition 1.2.1.3.1) originally presented in [5] as a composition of Euclidean string operations. This generalization will buttress an alternate proof that Euclidean strings characterize class one ERs in integer necklace form (See Corollary 2.2.11).

A significant result that will be used in the proof of Proposition 2.1.4 of this paper pertains to Fibonacci strings. Upon defining the morphisms $b \rightarrow a$ and $a \rightarrow ab$, it follows that when $a = 0$ and $b = 1$ the set of Fibonacci strings is given by $\{b, a, ab, aba, abaab, abaababa, ...\}$. It is proven that each Fibonacci string is a rotation of a binary Euclidean string. This implies that Fibonacci strings define binary necklaces that are class one ER.

1.2.1.3. "*The Distance Geometry of Music*" - Demaine et al. [5]

[5] does much work to synthesize the objects that are considered class one ERs, however much of this work had already been done by [4] for which [5] gives credit. [5] provide two novel results to the literature on class one ERs: firstly, a new algorithm that generates class one ERs is defined, and secondly a geometric definition of maximal evenness is given and proven to be equivalent to Definition 1.1.3.1. The new algorithm presented in this paper is equivalently summarized as follows:

Definition 1.2.1.3.1. (Demaine[12] Algorithm)

Given $k, n \in \mathbb{Z}^+$ such that $k \geq n$, let $r = k \bmod n$, and Demaine$(n,r) = (x_0, x_1, \ldots, x_{r-1})$ then

1. If n divides k, then Demaine(k,n) = $\left(\left(\frac{k}{n}\right)^n\right)$, where juxtaposition denotes concatenation and exponentiation denotes the number of juxtapositions.
2. If n does not divide k, then Demaine(k,n) = $\left(\left\lfloor\frac{k}{n}\right\rfloor^{x_0 - 1}\left\lceil\frac{k}{n}\right\rceil\left\lfloor\frac{k}{n}\right\rfloor^{x_1 - 1}\left\lceil\frac{k}{n}\right\rceil \ldots \left\lfloor\frac{k}{n}\right\rfloor^{x_{r-1} - 1}\left\lceil\frac{k}{n}\right\rceil\right)$.

The concatenation of these two strings, in either order, produces the original string. We can proceed to find the Euclidean strings that concatenate to form $E_{14,25}$ by looking at the neighbouring reduced fractions of 14/25 in the Farey sequence of order 25. Similarly for $E_{5,9}$. Since all Euclidean strings can be decremented by Definition 1.2.1.2.1b, they can be transformed such that the smaller integer is 0 and the larger is 1, implying that every Euclidean string can be "decremented" to a binary Euclidean string with the above concatenation property. Simply applying the increment operation by the amount decremented, the original Euclidean string is obtained. Therefore, every Euclidean string is the concatenation of smaller Euclidean strings.

[12] In [5] this algorithm is called "EUCLIDEAN", but this name is already attributed to the division algorithm, which is needlessly confusing, so it is here referred simply by the name of the leading author of [5].

It is proved that the Demaine algorithm produces class one ERs in integer rhythm form. No investigation was done to determine what structural insights the Demaine algorithm can reveal about class one ERs in [5], however one such structural result will be presented in Lemma 2.1.3 of this paper. [5] presents a geometric definition of a general evenness measure as well as maximal evenness on rhythms:

Definition 1.2.1.3.2. (Geometric Evenness of a Rhythm):
Let R be a rhythm with *k* time units and *n* event occurrences arranged on a circle with adjacent time units equidistant apart. The evenness of R is the sum of all Euclidean distances between each adjacent time unit with an event occurrence.

Definition 1.2.1.3.3. (Geometric Maximal Evenness):

R is maximally even if the geometric evenness of R reaches its maximum for its number of time units and event occurrences.

The main result of [5], here presented in equivalent terms, regarding maximal evenness is a theorem showing that the following are equivalent for a rhythm R with *k* time units and *n* event occurrences: [13]

1. R satisfies Definition 1.2.1.3.3.
2. R is the rhythm derived from the Clough-Douthett algorithm.
3. R is the rhythm derived from the Demaine algorithm.
4. R is a class one ER.

The proofs that geometric maximally even rhythms are class one ERs and the Demaine algorithm produces class one ERs are novel contributions of this paper, however most of the remaining maximal evenness proofs are alternative proofs of already established results from [4] presented in more standard mathematical notation. For instance, an alternate proof of the uniqueness of class one ERs for any *k* and *n* such that $n \leq k$ is given.

1.2.1.4 "*Interlocking and Euclidean Rhythms* – Gomez-Martin et al. [6]

ERs are defined in this paper as the binary rhythm derived from Bjorklund's algorithm, however, throughout their proofs ERs are characterized as the subset class rhythms calculable from the Clough-Douthett algorithm. Therefore, while ERs are here defined as class two, the results of this paper only apply to class one ERs because class one and two ER equivalency is assumed. This paper reproves [4]'s

[13] There is another equivalency in [5]'s theorem, regarding Clough-Douthett's "SNAP" algorithm, however it was already proved in [4] that this algorithm produces class one ERs.

complementation result for class one ERs, and their novel results pertain to identifying weight and length constraints of musically motivated operations closed on class one ERs. These operations are complementation, alternation, and decomposition. A brief discussion regarding the musical application of these operations will be made to illustrate a motivation for studying musical rhythm evenness.

Recall that [4] defined the complement of a subset class rhythm ME(k,n) as the collection of sets that are set complements to each M of ME(k,n) in {0, 1, …, k - 1}, and it was shown that this complement collection was ME(k,$k – n$). [6] uses an equivalent definition of complementation which is to consider the binary representation of a rhythm and switch all 1s for 0s and all 0s for 1s. They then provide an alternate proof of Clough and Douthett's result that complementation preserves rhythm maximal evenness and show explicitly that the complement of $ER_{k,n}$ is $ER_{k,k-n}$.

After the complement operation, [6] discusses an operation called "alternation", however alternation only applies to particular rotations of class one ERs The "j-alternation of order c" is an operation that, beginning at a j^{th} entry of $ER_{k,n}$, transforms, cycling around $ER_{k,n}$ without reaching j again, all c multiples of 1s from j into 0s. That is, every 1 that is a multiple of c 1s from entry j is switched to a 0, where the multiples of c are less than j. The main result regarding the alternation operation is that all alternations of order c of a class one ER with weight k and length n are class one ER if and only if c divides n. The third operation discussed in this paper is "union", which is defined as performing disjunction on the 1s of two binary rhythms with the same length. The result of union performed on such a pair of rhythms is another rhythm of the same length with the 1s in the places where 1s existed in either of the rhythms operated on. A rhythm resulting from the union of a pair of rhythms can be "decomposed" into this original pair of rhythms; this result is stated formally as follows: A class one ER with weight k and length n where $\left\lfloor \frac{k}{2} \right\rfloor < n < k$ is the union of rotations of two disjoint class one ERs $ER_{k-n,k}$ and $ER_{2k-n,n}$.

The application of these operations can be found in the compositional technique of the "rhythmic canon" whereby multiple and distinct rhythms with the same number of time units are layered overtop one another in such a way that each of the k time units in the song contains an event occurrence from one and only one of the rhythms. In the context of musical rhythms, "event occurrences" are times when a sound is made. Therefore, having well defined operations like complementation, alternation and decomposition that exclude sound events in such a way that evenness is preserved is useful to a composer using the rhythmic canon technique. Since maximal evenness appears to be an important property of rhythms [7], it would be interesting to determine how well these operations preserve general, potentially non-maximal rhythm evenness as measured by an evenness measure perhaps related to Bjorklund's metric[14]. Enabling composers to classify rhythms by evenness could help composers design rhythmic canons explicitly in terms of the contrasting evenness values of the rhythms in each layer of their compositions.

1.2.2.0 Class Two ERs:

Recall that class two ERs are characterized simply as the binary necklace forms of the results of the Bjorklund algorithm. Most of the structural investigation of ERs pertains to class one ERs because the two classes are assumed in the literature to be equivalent. Little is known about the structure of class two ERs, because they are largely studied as if they are class one. There are two main papers that define Euclidean rhythms by the Bjorklund algorithm. The first paper does not have mathematical results; however, it is the first to tie the term Euclidean to the notion of maximal evenness. The second paper is the only paper to present structural insight into class two ERs, however most of its results are based on the class one and two ER equivalency assumption. So, unfortunately most of their results do not apply to either class of ERs until a proof of equivalency is provided.

1.2.2.1. "*The Euclidean Algorithm Generates Traditional Musical Rhythms*" - Toussaint [7]

The first consideration of maximal evenness as applied to musical rhythms appears to begin with [7] in which the term Euclidean rhythm was first used. This paper is more ethnomusicological than mathematical. It briefly describes Bjorklund's algorithm using an example and points out that the calculated sequence is intuitively maximally even. Then, using the same two parameters from the Bjorklund algorithm example, the greatest common divisor of these parameters is calculated using the Euclidean algorithm. A relationship between the values at each step of the two algorithms is noted and so the class of binary necklaces resulting from Bjorklund's algorithm is here named Euclidean rhythms. This means that while class two ERs were called Euclidean rhythms before class one ERs. After defining this class of rhythms, the paper concludes with a detailed discussion of the prevalence of these rhythms as musical rhythms generally, but particularly in world music. It should be noted that Toussaint does not use a precise definition of maximal evenness in his discussion; he uses the characterization of "calculation by Bjorklund's algorithm" as sufficient characterization of maximal evenness here.

1.2.2.2. "*Structural Properties of Euclidean Rhythms"* - Gomez-Martin et al. [3]

In [3], Euclidean rhythms are defined as class two ERs. While [3] cites [5] as showing that the Bjorklund and Clough-Douthett algorithms produce equivalent rhythms, no proof of this fact actually exists in [5] – only a few examples are provided. Since the Clough-Douthett algorithm was shown by [4] and [5] to produce class one ERs and Bjorklund's algorithm produces class two ERs by definition, [3]

[14] The term metric in Bjorklund's metric is used informally. It is not technically a metric and is instead more similar to a measure.

assumes the intuitively true equivalency between the two ER classes. Most of their results are based on the equivalency assumption and so will not be discussed in depth because they do not apply formally to either class of ER unless their assumption can be proven. Fortunately, [3] does provide two salient structural results about class two ERs by appealing to the structure of the Bjorklund algorithm. One of these structural results will be used in section 2 to prove class one and two ER equivalency, and so will be discussed here. The following two equations will be necessary for our investigation:

Definition 1.2.2.2.1 (Length and Weight Functions):

Let A be the set of all integer necklaces, then for every $a \in A$, define $L, W \to \mathbb{Z}$ given by the following equations:

a. $L(a) = |a|$,

b. $W(a) = \sum_{i=0}^{L(a)-1} a_i$.

$L(a)$ is called the length of a, and $W(a)$ is called the weight of a.

The first main result of [3] is the insight that Bjorklund's algorithm produces a necklace consisting of the concatenation of a repeated "main" pattern and, when $(k,n) = 1$, an unrepeated "tail" pattern. If $(k,n) > 1$, then the tail pattern does not exist. By Bjorklund's algorithm, the columns of the matrix A at the end of the algorithm are all identical to each other but different from B. [3] defines the main pattern M to be a column of A and the tail pattern T to be B, where p is the number of repetitions of M. Therefore $B_{k,n} = M^pT$.[15] The following basic equations follow from this concatenation structure of class two ERs:

Definition 1.2.2.2.2 (Equations from [3])

Let $B_{k,n} = M^pT$ be a class two ER. Then,

a. $k = W(M) \cdot p + W(T)$,

b. $n = L(M) \cdot p + L(T)$,

$W(T) = L(T) = 0$ when $(k, n) > 1$.

The second important result of [3] is their first lemma which can be restated as follows:

Lemma 1.2.2.2.3 (Gomez-Martin et al. lemma)

Let $B_{k,n}$ be a class two ER where $1 \leq n < k$. Then the following equations hold:

a. If $(k, n) > 1$ then $k \cdot L(M) - n \cdot W(M) = 0$.

[15] Here, juxtaposition denotes concatenation.

b. If $(k, n) = 1$ then $k \cdot L(M) - n \cdot W(M) = \pm 1$, and $W(T) \cdot L(M) - L(T) \cdot W(M) = \pm 1$. Lemma 1.2.2.2.3 is used extensively under the assumption that it also applies to class one ERs to prove most of the results in [3].

The following results are based on class one and two ER equivalence and pertain to the properties of these main and tail patterns of ERs. They show that both the main and tail patterns are ERs and that the main pattern cannot be formed as the concatenation of repetitions of a smaller string, that is, the main pattern is minimal. An unstated corollary of Definition 1.2.2.2.1. and facts about the Bjorklund algorithm show a useful result about class two ERs: when $(k,n) > 1$, $B_{k,n}$ is (k,n) concatenations of $B_{k/(k,n),\, n/(k,n)}$. This result will be proven in Lemma 2.1.2 of this paper.

1.3. Conclusion

It appears that the literature is unaware that Euclidean rhythm refers to two distinct classes of rhythms: those that have been shown to satisfy Definition 1.1.3.1., and those that are results of the Bjorklund algorithm. It is assumed that these two ER classes are equivalent, however no proof of this equivalency exists. A proof of their equivalency is given in the proceeding section of this paper.

As illustrated from the discussion in section 1.2.1.1. on [4], there appears to exist interesting and subtle results about maximal evenness in the mathematical music theory literature that have not been explored in the more recent more mathematical literature. A deeper investigation into the mathematical music theory literature may provide more insights into rhythm maximal evenness. Both [3] and [1] provided insight into the concatenation structure of class two and class one ERs, respectively. If class one and two ERs are indeed equivalent, then comparing these two concatenation structures could reveal that the two structures are also equivalent.

There are several algorithms that all construct ERs: Bjorklund, Demaine, Clough-Douthett, and others. For the sake of synthesis, it would be worthwhile to study these algorithms and compare their implications on structural properties of ERs. To this end, Lemma 2.1.2 and Lemma 2.1.3 of the proceeding section are such contributions; and in Proposition 2.1.4. it is shown that the concatenation structure of ERs described by [3] is identical to another concatenation structure, first described in this paper, derived from the Demaine algorithm.

As [7] [5], and [8] all discuss, Euclidean rhythms are highly prevalent in music generally. Also, a cursory internet search on Euclidean rhythms reveals that they are of interest to many for their utility in music writing. This popularity indicates the importance of having a detailed and parsimonious mathematical understanding of Euclidean rhythms that can facilitate their more sophisticated application to music. Since the literature is unclear precisely what a Euclidean rhythm is, and little work has been done to synthesize facts known about the different maximally even necklaces (Euclidean strings, ME sets, Bjorklund rhythms, etc), further synthesis work on Euclidean rhythms is recommended.

2. Mathematical Results

2.1. Proof of Class One and Two ER Equivalence

2.1.0. Introduction

It appears that no mathematical proof exists showing that class two ERs are maximally even. In the paper that originally presents the Bjorklund algorithm, Bjorklund appeals to the reader's intuition that his algorithm presents maximally even sequences by showing examples that appear to have some sort of maximally even property, and no formal definition of maximal evenness is presented. Recall that [3] cite [5] as showing that class two ERs are equivalent to the class one ERs resulting from the Clough-Douthett algorithm, however [5] show nothing more than examples of this fact. As mentioned in the review, this means that most of the results of [3] apply if and only if class one and two ERs can be proven to be equivalent. Below is my proof of this fact.

2.1.1. Proof Sketch

This proof begins with two lemmas derived from a structural analysis of the Bjorklund and Demaine algorithms, which generate class two and class one ERs, respectively. It is then shown that class two ERs are constructed as concatenations of Fibonacci strings, which Ellis et al. showed to be rotations of Euclidean strings and therefore class one ERs. Using the structural lemmas and uniqueness of class one and two ERs, the Fibonacci string concatenation structure is shown to imply the equivalency of the two ER classes.

Lemma 2.1.2. (Concatenation property of class two ERs)

When $d = (k, n) > 1$, $B_{k,n} = \left(B_{\frac{k}{d}, \frac{n}{d}}\right)^d$.

Proof:

When $(k, n) > 1$, [Gomez-Martin et al. 1] showed that $B_{k,n}$ is the concatenation of (k, n) repetitions of some weight $\frac{k}{(k, n)}$ and length $\frac{n}{(k, n)}$ rhythm M. Since $B_{k,n} = M^{(k,n)}$, we want to show that $M = B_{\frac{k}{(k,n)}, \frac{n}{(k,n)}}$. The correspondence between the Euclidean and Bjorklund algorithms is that the number of columns of A and B at each step of the Bjorklund algorithm are adjacent remainders in the calculation of (k,n). So it follows from the Euclidean algorithm and this correspondence that when calculating $B_{k,n}$, the number of columns of A and B will be (k,n) times that when calculating $B_{k/(k,n),\, n/(k,n)}$. Also, since the number of 1s and 0s for $B_{k,n}$ is just (k,n) times that of $B_{k/(k,n),\, n/(k,n)}$, it follows that the columns of A and B for each calculation are identical until the step t when $B_{k/(k,n),\, n/(k,n)}$ is determined. It follows that when calculating $B_{k,n}$ with $r_{s+1} = \left(\frac{k}{(k, n)}, \frac{n}{(k, n)}\right) = 1$, and r_s is the previous remainder in the division algorithm for $\left(\frac{k}{(k,n)}, \frac{n}{(k,n)}\right)$, the A and B matrices at step t of the Bjorklund algorithm (where $B_{\frac{k}{(k,n)}, \frac{n}{(k,n)}}$ is calculated in t steps) are $A = \left[\left(A_t\right)^{r_s \cdot (k,n)} \cdots \left(A_t\right)\right]$ and $B = \left[\left(A_{t-1}\right)^{(k,n)} \cdots \left(A_{t-1}\right)\right]$,

implying that the next step in calculating $B_{k,n}$ has A and B matrices: $A = \begin{bmatrix} \left(A_t\right)^{(k,n)} \cdots \left(A_t\right) \\ \left(A_{t-1}\right)^{(k,n)} \cdots \left(A_{t-1}\right) \end{bmatrix}$ and

$B = \left[\left(A_t\right)^{\left(r_s - 1\right) \cdot (k,n)} \cdots \left(A_t\right)\right]$. This pattern will repeat r_s times until B is empty, but A is here constructed such that there will always be (k,n) columns of A, and the main pattern of $B_{k,n}$ will be

$A_t^{r_s} \cdot A_{t-1} = B_{\frac{k}{(k,n)}, \frac{n}{(k,n)}}$. Therefore, the main pattern of $B_{k,n}$ when $(k,n) > 1$ is repeated (k,n) times, and is $B_{\frac{k}{(k,n)}, \frac{n}{(k,n)}}$.

□

<u>Lemma 2.1.3. (Class one ERs have Similar Concatenation Scheme as Class two ERs)</u>

Let $r_{-1} = k$, $r_0 = n$, and $r_i = r_{i-2} \bmod r_{i-1}\ \forall\, i \in [1, s+1]$, where $r_{s+1} \in \{1, 0\}$. And define the following:

$f_i = \left\lfloor \frac{r_i}{r_{i+1}} \right\rfloor \forall\, i \in [-1, s-1]$, with $f_s = r_s$, and $c_i = \left\lceil \frac{r_i}{r_{i+1}} \right\rceil \forall\, i \in [-1, s-1]$. Then there exist class one ERs V and U such that

If $(k,n) = 1$, then $ER_{k,n} = V^{f_s} U$, and

If $(k,n) > 1$, then $ER_{k,n} = V^{(k,n)} = \left(ER_{\frac{k}{(k,n)}, \frac{n}{(k,n)}}\right)^{(k,n)}$.

<u>Proof:</u>

By the Demaine algorithm, when $(k,n) = 1$:

$$
\begin{aligned}
&E_1 = Demaine\left(r_s, r_{s+1}\right) = f_s \\
&\rightarrow E_2 = Demaine\left(r_{s-1}, r_s\right) = \left(f_{s-1}\right)^{f_s - 1} \cdot c_{s-1} \\
&\rightarrow E_3 = Demaine\left(r_{s-2}, r_{s-1}\right) = \left(\left(\mathrm{V}^*\right)^{f_s - 1}\right) \cdot \left(\left(f_{s-2}\right)^{c_{s-1} - 1} \cdot c_{s-2}\right) = \left(\left(V^*\right)^{f_s - 1}\right) \\
&\qquad \cdot U^* \cdot V^* \sim_{rot} U^* \cdot \left(V^*\right)^{f_s}
\end{aligned}
$$

,

where $V^* = \left((f_{s-2})^{f_{s-1}-1} \cdot c_{s-2} \right)$, and $U^* = f_{s-2}$. That is, E_3 is the concatenation of f_s repetitions of V^* and a U^* .

When $(k,n) > 1$, we apply step 1 of the Demaine algorithm to initialize at E_2 because $E_1 = Demaine((k,n),0) = \varnothing$, therefore:

$E_2 = Demaine\left(r_{s-1}, r_s\right) = f_{s-1}^{f_s}$, because $r_s \mid r_{s-1}$.

$\rightarrow E_3 = Demaine\left(r_{s-2}, r_{s-1}\right) = \left(V^*\right)^{f_s} \cdot U^*$, where $V^* = \left((f_{s-2})^{f_{s-1}-1} \cdot c_{s-2} \right)$, and $U^* = \varnothing$. Notice that in both cases of (k,n) , V^* is equivalent, however U^* is empty when $(k,n) > 1$.

If we keep applying this algorithm to the end on each E_3 until we get to $E_s = Demaine\left(r_{-1}, r_0\right)$, it follows that the $\left(V^*\right)$ s generate the same sequence of ceiling and floor elements at each application and U^* similarly generates its own sequence of ceiling and floor elements (unless it is empty). Define V to be the sequence of ceiling and floor elements in E_s generated from V^* , and U to be the sequence of ceiling and floor elements of E_s generated from U^* . Therefore, since E_s is a class one ER of weight *k* and length *n* by [5], it follows that $ER_{k,n}$ is the concatenation of f_s V patterns and one U pattern. It also follows by the Demaine algorithm that both V and U must be class one ERs themselves because both V^* and U^* are class one ERs and the Demaine algorithm preserves the class one ER characterization at each step as shown by [5]. What is more, since U is empty when (*k*,*n*) > 1, V must have weight *k*/(*k*,*n*) and length *n*/(*k*,*n*); so, since class one ERs are unique for a given weight and length, when $d = (k,n) > 1$,

$ER_{k,n} = (V)^d = \left(ER_{\frac{k}{d}, \frac{n}{d}}\right)^d$.

□

<u>Proposition 2.1.4 (Class one and class two ERs are Equivalent)</u>

Let $ER_{k,n}$ and $B_{k,n}$ be the class one and two ERs of weight k and length n . Then $ER_{k,n} = B_{k,n}$.

<u>Proof:</u>

Part 1 (The main and tail patterns of class two ERs are class one ERs):

Let $B_{k,n}$ denote a class two ER of weight *k* and length *n*. By [3], $B_{k,n}$ is the concatenation of a main pattern M repeated p times and an unrepeated tail pattern T . That is, $B_{k,n} = M^p \cdot T$. By the definition of the Bjorklund algorithm, after the final step, the main pattern is a column of the matrix A, and the tail pattern is the column of matrix B and is empty when $(k, n) > 1$. Since [1] showed that Fibonacci strings are rotationally equivalent to Euclidean strings, which are particular rotations of class one ERs, the proof will proceed by showing that the columns of A are Fibonacci strings. The columns of A are Fibonacci strings in the following way:

Let $(k, n) = 1$.

Since when $n > \left\lfloor \frac{k}{2} \right\rfloor$, the 1s and 0s of the binary rhythm form of $B_{k,n}$ are switched, which has no effect on whether the rhythm is maximally even, WLOG, let $n \leq \left\lfloor \frac{k}{2} \right\rfloor$.

Let A_i denote a column of A at step i of the Bjorklund algorithm following the initial steps of the algorithm. For each column of A, the initial two steps of the Bjorklund algorithm repeatedly add 0 bits to a 1 until B contains no more bits such that the result is $A_2 = 1 \cdot 0^{\left\lceil \frac{k}{n} \right\rceil - 1}$, where $A_1 = 1 \cdot 0^{\left\lfloor \frac{k}{n} \right\rfloor - 1}$. Therefore, for $i \geq 2$, the columns of B contain no bits, and will henceforth contain only columns of A from the previous step. This means that for $i \geq 3$, $A_i = A_{i-1} \cdot B_{i-1} = A_{i-1} \cdot A_{i-2}$. The Fibonacci string is found in the following way: let $c = \delta(A_2)$ and $f = \delta(A_1)$, then the sequence $\{\delta(A_i) : i \in [1, t]\} = (f, c, cf, cfc, cfccf, cfccfcfc, cfccfcfccfccf, \ldots, A_t)$, (where A_t is the main pattern of $B_{k,n}$) is a sequence of Fibonacci strings given by the morphisms $f \rightarrow c$ and $c \rightarrow cf$. Recall that Ellis et al. showed that when *f* = 0 and *c* = 1, all strings resultant from any number applications of the above morphisms is a Fibonacci string rotationally equivalent to a Euclidean string, it follows that for any *i*th Fibonacci string, if the *f* and *c* are incremented by $\left\lfloor \frac{k}{n} \right\rfloor$, we get $\delta(A_i)$. Since Ellis et al. showed that the increment operation is closed on Euclidean strings, and since $c = f + 1$, it follows that each $\delta(A_i)$ defines a rotation of a Euclidean string. Since Euclidean strings were shown by Ellis et al. to be class one ERs, and δ^{-1} preserves rotational equivalence equivalence to a Euclidean string, the columns of A at each step of the Bjorklund algorithm for any k and n are class one ERs. Therefore, the main and tail patterns of $B_{k,n}$ are class one ERs because they are both defined as columns of A from the Bjorklund algorithm.

Part 2 (Class one and two ERs are equivalent):

To recap, $B_{k,n} = M^p \cdot T$ and $ER_{k,n} = V^{f_s} \cdot U$, where all of $M, T, V,$ and U are class one ERs. To finish the proof, I will show that $p = f_s$ for any k and n and then use this fact to show that $V = M$ and $U = T$.

Consider the remainder and floor sequences defined in Lemma 2.1.3, the terms will be used again here:

Case 1. $d > 1$:

Since $f_s = r_s$ where $r_{s+1} = 0$ it follows by the division algorithm that $f_s = d.$ And $p = d$ when $d > 1$ by the Bjorklund algorithm as shown by [3].

<u>Case 2. $d = 1$:</u>

Recall that at the end of each subtraction step in the Bjorklund algorithm, the number of columns of A and B are adjacent remainders in the Euclidean algorithm. So, when $(k, n) = 1$, T is defined as the single column in B of the final subtraction step of the Bjorklund algorithm, that is, B has $r_{s+1} = 1$ columns, and the number of columns of A at the end of the final subtraction step is r_s , which is equal to f_s . Since p is defined as the number of columns of A after the final subtraction step of the Bjorklund algorithm, $p = f_s$. Therefore, in any case $p = f_s$.

Let $(k, n) = 1$, $B_{k,n} = M^p T$. Then M and T are class one ERs. By the equations from Definition 1.2.2.2.2, and uniqueness of class one and two ERs, for any k and n , $B_{k - W(M) \cdot p,\, n - L(M) \cdot p} = T$ and there exists class one ER U such that $ER_{k - W(M) \cdot p,\, n - L(M) \cdot p} = U$, implying that both U and T have the same weight and length and so are the same class one as well as class two ER.

Similarly for M and V: We have $B_{\frac{k - W(T)}{p},\, \frac{n - L(T)}{p}} = M$, and so there exists class one ER V such that $ER_{\frac{k - W(T)}{p},\, \frac{n - L(T)}{p}} = V$, implying that both M and V have the same weight and length and are both class one and two ERs. This implies that $W(V) \cdot p = W(M) \cdot p \Rightarrow W(V) = W(M)$, and $L(M) \cdot p = L(V) \cdot p \Rightarrow L(M) = L(V)$. So, by uniqueness of both classes of ERs, $B_{k,n} = M^p T = V^p U = ER_{k,n}$.

Let $d = (k, n) > 1$, then by Lemma 2.1.2, $B_{k,n} = \left(B_{\frac{k}{d}, \frac{n}{d}}\right)^d$, and since $\left(\frac{k}{d}, \frac{n}{d}\right) = 1$, it follows by the above paragraph and Lemma 2.2.3, that $B_{k,n} = \left(ER_{\frac{k}{d}, \frac{n}{d}}\right)^d = ER_{k,n}$.

So, class one and two ERs are equivalent.

□

<u>2.2. Generalization of the Demaine Algorithm</u>

<u>2.2.1. Introduction</u>

This section has two objectives: generalizing the Demaine algorithm and providing an alternate proof that Euclidean strings characterize class one ERs. First, a generalization of the Demaine algorithm will be made in terms of operations defined in [1]. This generalization applies the same operation of the Demaine algorithm at each step, however the Demaine algorithm is applied only to Euclidean rhythms with parameters from the remainder sequence calculated from the division algorithm. The generalization presented here is defined for all sequences. To construct this generalization, the concept of a dichotomous sequence will be defined. Using the constructions to achieve the first objective, the second objective of providing an alternate proof that Euclidean strings are class one ERs will be made.

2.2.2. Preliminary notes

Throughout this section, the following hold:

1. Let k and n be relatively prime positive integers such that $k > n$, where $r = k - n\left\lfloor \frac{k}{n} \right\rfloor$, and d is the integer such that $kd \equiv 1 \bmod(n)$.
2. All sequence indices are modulo the length of the sequence.
3. $\sim_{rot}$ denotes the rotational equivalence relation between two sequences.

Definition 2.2.3. (Dichotomous Sequence)

A length n integer sequence $a = (a_0, ..., a_{n-1})$ is called a dichotomous sequence iff there exists an integer f^a such that the elements of a come only from the set $\{f^a, f^a + 1\}$ where f^a and $f^a + 1$ are called the floor and ceiling elements of a, respectively. For brevity, denote $f^a + 1$ by c^a. Define the floor index set of a, denoted F^a, to be the set of indices of a at which there is a floor element; similarly define C^a to be the ceiling index set of a. Note that $C^a = \{0, 1, ..., n-1\} \setminus F^a$ for all dichotomous sequences a with length n.

Definition 2.2.4. (Characterization of Euclidean String in Terms of Dichotomous Sequences)

A Euclidean string is an integer sequence of unit length or a dichotomous sequence with weight k and length n, denoted $E_{k,n}$, with floor element $f^{E_{k,n}} = \left\lfloor \frac{k}{n} \right\rfloor$, where $C^{E_{k,n}} = \{(n - 1 + jd) \bmod n : j \in [0, r-1]\}$, and $F^{E_{k,n}} = \{jd \bmod n : j \in [0, n-r-1]\}$.

Definition 2.2.5. (T Operation)

Let a be a dichotomous string and f an integer. Then with $c = f + 1$, define the operation:

$$T(a,f) = Inc^f(\delta(a)) = \left(f^{(a_0 - 1)}, c, f^{(a_1 - 1)}, c, ..., f^{(a_{n-1} - 1)}, c\right).$$

$T(a,f)$ is therefore a dichotomous string with floor element f.

Observation 2.2.6. (T Generalizes the Demaine Algorithm)

The T operation is a generalization of the Demaine algorithm in [5] to any dichotomous sequence and floor element input. That is, let $a = Demaine(n, r)$, then $Demaine(k, n) = T\left(a, \left\lfloor \frac{k}{n} \right\rfloor\right)$.

Observation 2.2.7. (More Weight and Length Equations)

Let i be a positive integer and a be a dichotomous sequence. Then the following equations hold:

(a). $W(\delta(a)) = L(a)$

(b). $L(\delta(a)) = W(a)$

(c). $W\left(Inc^i(a)\right) = L(a) \cdot i + W(a)$

(d). $L\left(Inc^i(a)\right) = L(a)$

Putting these Equations together, it is clear that

(e). $W(T(a, i)) = W(a) \cdot i + L(a)$

(f). $L(T(a, i)) = W(a)$.

Explanation:

(a,b) Follow directly since δ turns all elements a_j of *a* into a sequence of $(a_j - 1)$ 0s followed by a single 1.

(c) and (d) are obvious from the definition of Inc^i .

Lemma 2.2.8. (*T* Preserves Rotational Equivalence to Euclidean Strings)

$T\left(E_{n,r}, \left\lfloor \frac{k}{n} \right\rfloor\right)$ and $E_{k,n}$ are rotationally equivalent. Equivalently, the Demaine algorithm preserves rotational equivalence to Euclidean strings.

Proof:

The proof follows from the work of (Ellis et al.) who showed in their Theorem 2 that $\delta\left(R\left(E_{n,r}\right)\right) \sim_{rot} E_{r,n}$, and in their Lemma 4 that $R\left(E_{r,n}\right) \sim_{rot} E_{r,n}$. Therefore

$T\left(E_{n,r}, \left\lfloor \frac{k}{n} \right\rfloor\right) = Inc^{\left\lfloor \frac{k}{n} \right\rfloor}\left(\delta\left(E_{n,r}\right)\right) \sim_{rot} Inc^{\left\lfloor \frac{k}{n} \right\rfloor}\left(E_{r,n}\right)$, and since $k = r + \left\lfloor \frac{k}{n} \right\rfloor n$, it follows by (Observation 2.2.7.e) that $Inc^{\left\lfloor \frac{k}{n} \right\rfloor}\left(E_{r,n}\right) = E_{r + \left\lfloor \frac{k}{n} \right\rfloor n,\, n} = E_{k,n}$.

□

Definition 2.2.9. (*T* Recursion on a Dichotomous Sequence)

Let *a* be a dichotomous sequence, and *q* be a sequence of positive integers with length *s*, then define the following recursive algorithm:

$T^1(a, q) = T\left(a, q_1\right)$, and

$T^j(a, q) = T\left(T^{j-1}(a, q), q_j\right), \forall j \in [2, s]$

That is, $T^s(a, q)$ applies *T* recursively to the dichotomous sequence *a s* times with the proceeding integer in *q* as the new floor element and the sequence resulting from the previous application of *T* as the new dichotomous sequence. This algorithm generates a class of dichotomous sequences from a floor element sequence and either a unit length integer sequence or a dichotomous sequence.

Example:

Let $a = (3)$ be the initial sequence, and $q = (2, 1, 3)$ be a floor element sequence. Then

Let $a = (3)$ be the initial sequence, and $q = (2, 1, 3)$ be a floor element sequence. Then

$T^3(a, q) = T\left(T^2(a, q), 3\right) = T(T(T(a, 2), 1), 3) = T(T((2, 2, 3), 1), 3) = T((1, 2, 1, 2, 1, 1, 2),$
$3) = (4, 3, 4, 4, 3, 4, 4, 4, 3, 4)$

Proposition 2.2.10. (Euclidean String Rotational Equivalence Given Floor Element Sequence)

Let $r_{-1} = k, r_0 = n,$ and $r_i = r_{i-2} \bmod r_{i-1}$ for all $i \in [1, s]$ where $r_{s+1} = 1$. Now, define $q = \left(\left\lfloor \frac{r_{s-i-1}}{r_{s-i}} \right\rfloor : i \in [0, s]\right)$. That is, q is the sequence of quotients of the $s + 1$ step Euclidean algorithm to calculate (k, n) from last step to first. Then $T^i(r_s, q) \sim_{rot} E_{r_{s-i-1}, r_{s-i}}$, for all $i \in [1, s]$.

Proof:

Base Case ($i = 1$):

Since (r_s) is of unit length, $(r_s) = E_{r_s, 1}$, let $r = 1, n = r_s, k = r_{s-1}$, then $T\left(E_{n, r}, \left\lfloor \frac{k}{n} \right\rfloor\right) \sim_{rot} E_{k, n}$, by (Lemma 2.2.8.).

Inductive Step ($i \in [2, s]$):

By (Definition 2.2.9.) and the inductive hypothesis, $T^i(r_s, q) = T\left(T^{i-1}(r_s, q), q_i\right) \sim_{rot} T\left(E_{r_{s-i}, r_{s-i+1}}, q_i\right)$.

Now Let $r = r_{s-i+1}, n = r_{s-i},$ and $k = r_{s-i-1}$. Then by (Lemma 2.2.8.), we have $T\left(E_{n, r}, \left\lfloor \frac{k}{n} \right\rfloor\right) \sim_{rot} E_{k, n}$.

□

Corollary 2.2.11. (Euclidean strings are maximally even)

Since T preserves rotational equivalence to Euclidean strings and the Demaine algorithm is a special case of the T operation, they both calculate the same sequence from the same initial parameter (a unit length Euclidean string with weight r_s). It follows that the sequences resulting from the Demaine algorithm are rotations of Euclidean strings. Since [5] proved that the Demaine algorithm characterizes class one ERs, it follows by rotational equivalence that Euclidean strings characterize class one ERs as well.

3. Rotationally Invariant Compositions

3.1. Definitions and Properties

3.1.0. Introduction

In this section, a collection of rotationally invariant compositions, equivalent to integer necklace rhythms, will be constructed with some of their basic properties described. The work in this part is meant to provide a groundwork from which to pursue further study.

3.1.1. Constructing These Compositions

Below is a simple but efficient four step construction of rotationally invariant compositions in terms of partitions:

1. Let k be a positive integer, then a partition of k is defined as a multiset of integers that sum to k .
2. Let P_k be the set of partitions of k , then a composition of k is defined as a sequence of all the elements in some $p \in P_k$.
3. Let C_k^* be the set of compositions of k , then define $C_{k,n}^* \subset C_k^*$ to be the set of compositions of k with length n .
4. Define $C_{k,n} = \dfrac{C_{k,n}^*}{\sim_{rot}}$ to be the set of rotationally invariant compositions of k with length n .

   Similarly define $C_k = \dfrac{C_k^*}{\sim_{rot}}$.

Observation:

Notice that the *n* elements of a composition in $C_{k,n}$ must sum to *k*; since these compositions are rotationally invariant, it follows that $C_{k,n}$ is equivalent to the set of all rhythms of weight *k* and length *n*.

Definition 3.1.2. (Maximally Even Composition):

Let $a \in C_{k,n}$, then a is called maximally even iff $\forall\, i \in [0, n-1], \forall\, l \in [1, n], \sum_{j=i}^{i+l-1} I_j \in \left\{ \left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil \right\}$.

Definition 3.1.3. (Minimally Even Composition):

Let $a \in C_{k,n}$, then a is called Minimally even iff $\forall\, l \in [1, n], \forall\, i \in [0, n-1], \sum_{j=i}^{i+l-1} a_j \in \{l, k-n+l\}$.

Definition 3.1.4. (Inverse Pair):

Let $a, b \in C_{k,n}$, then a and b are called inverses iff $b = (a_n, a_{n-1}, \ldots, a_1) = a^{-1}$, where a^{-1} denotes the inverse of a .

Definition 3.1.5. (Palindromic Composition):

Let $a \in C_{k,n}$, then a is called a palindromic composition iff $a = a^{-1}$.

Definition 3.1.6. (Functions from [Ellis et al.]):

Let $a \in C_{k,n}$. Then define the following operations on $C_{k,n}$:

1. $\delta(a)$ applies to each element of a , the morphism $i \rightarrow 0^{i-1}1$, where i is an integer.
2. $\delta^{-1}(a)$ applies to each subsequence of a immediately following a one, the morphism $0^{i-1}1 \rightarrow i$.
3. $S(a)$ switches all ceiling and floor elements for the other, where a is a dichotomous sequence.

Definition 3.1.7. (Complement Function):

Let $a \in C_{k,n}$, then define $C(a) = \delta^{-1}(S(\delta(a))) \in C_{k,k-n}$. C is called the complement operation on $C_{k,n}$. Note that C is invertible because δ and S are invertible, so C defines a bijection between $C_{k,n}$ and $C_{k,k-n}$ implying that $|C_{k,n}| = |C_{k,k-n}|$.

Proposition 3.1.8. (Inverse Preservation):

Let $a \in C_{k,n}$. Then $C(a^{-1}) = C(a)^{-1}$

Proof:

$C(a) = \delta^{-1}(S(\delta(a)))$ is composed of inverse preserving operations:

$S(b^{-1}) = S(b)^{-1}$ is obvious for any binary necklace b and $\delta(a)$ is always a binary necklace.

δ Preserves Inverses:

$$\delta(a^{-1}) = \delta(a_{n-1}, a_{n-2}, ..., a_0) = \left(0^{a_{n-1}-1}, 1, 0^{a_{n-2}-1}, 1, ..., 0^{a_0-1}, 1\right) = \left(1, 0^{a_0-1}, 1, 0^{a_2-1}, ..., 1, 0^{a_{n-1}-1}\right)^{-1} = \left(0^{a_0-1}, 1, 0^{a_2-1}, 1, ..., 0^{a_{n-1}-1}, 1\right)^{-1} = \delta(a_0, a_1, ..., a_{n-1})^{-1} = \delta(a)^{-1},$$

δ^{-1} Preserves Inverses:

Let $b = \delta(a^{-1})$, then

$$\delta^{-1}(b^{-1}) = \delta^{-1}(b_{k-1}, b_{k-2}, ..., b_0) = \delta^{-1}\left(0^{a_{n-1}-1}, 1, 0^{a_{n-2}-1}, 1, ..., 0^{a_0-1}, 1\right) = a^{-1}$$

$$= (a_0, a_1, ..., a_{n-1})^{-1} = \delta^{-1}\left(0^{a_0-1}, 1, 0^{a_1-1}, 1, ..., 0^{a_{n-1}-1}, 1\right)^{-1} = \delta^{-1}(b)^{-1}$$

,

Therefore C preserves inverses.

□

Proposition 3.1.9. (Properties of Maximally and Minimally Even Composition):

(a): The maximally and minimally even elements of $C_{k,n}$ exist and are unique, and

(b): they are palindromic.

Proof:

(a):

By, since $C_{k,n}$ is the set of rhythms of weight k and length n, maximally even compositions are Euclidean rhythms by Definition 1.1.3.1, which exist and [5] showed are unique for any weight k and length n such that $n \leq k$. The existence and uniqueness of Euclidean rhythms has been proven by multiple authors in different ways.

By the definition of $C_{k,n}$, if for some $a \in C_{k,n}$ there is an $a_i = k - n + 1$, then the remaining $n - 1$ elements of a can and must all be 1 s. Therefore a exists and is unique.

(b):

Since the maximally even composition is a Euclidean rhythm, it is palindromic because when (*k*,*n*) = 1, it is rotationally equivalent to a Euclidean string, which is rotationally equivalent to its mirror image, i.e. it is a palindrome. When (*k*,*n*) > 1, the Euclidean rhythm is the concatenation of (*k*,*n*) Euclidean rhythms with weight *k*/(*k*,*n*) and length *n*/(*k*,*n*), which are all palindromes; but this means that the mirror image of the ER is still the concatenation of repetitions of the same palindromic ER. So, the maximally even composition is a palindrome.

Note that all but one of the elements of the minimally even composition are 1s, so it is impossible for it to be unequal to its inverse. That is, the minimally even composition is a palindrome.

□

<u>Proposition 3.1.10 (Bjorklund Metric Values):</u>

Let $a \in C_{k,n}$, and f be the Bjorklund metric (please see Definition 4.0.1. in the next section). Then a is maximally even $\Rightarrow \min\limits_{b \in C_{k,n}} \{f(b)\} = f(a)$,

<u>Proof:</u>

a is maximally even so $\forall\, i \in [0, n-1]$ and $\forall\, l \in [1, n]$, $\sum\limits_{j=i}^{i+l-1} a_j \in \left\{ \left\lfloor \frac{lk}{n} \right\rfloor, \left\lceil \frac{lk}{n} \right\rceil \right\}$, which means that it is always true that $\left(\left(\sum\limits_{j=i}^{i+l-1} a_j \right) - \frac{l \cdot k}{n} \right)^2 \in \left\{ \left(\left\lfloor \frac{lk}{n} \right\rfloor - \frac{lk}{n} \right)^2, \left(\left\lceil \frac{lk}{n} \right\rceil - \frac{lk}{n} \right)^2 \right\} < 1$. Since a is unique, for every other $b \in C_{k,n}$, it may be the case for some $l \in [1, n]$ that $\left(\left(\sum\limits_{j=i}^{i+l-1} b_j \right) - \frac{l \cdot k}{n} \right)^2 = \left(\left(\sum\limits_{j=i}^{i+l-1} a_j \right) - \frac{l \cdot k}{n} \right)^2 < 1$, however since b is not maximally even, there exists at least one $l^* \in [1, n]$ such that $\left(\left(\sum\limits_{j=i}^{i+l^*-1} b_j \right) - \frac{l^* \cdot k}{n} \right)^2 \geq 1$. So

$$f(a) = \sum_{i=0}^{n-1} \sum_{l=1}^{n} \left(\left(\sum_{j=i}^{i+l-1} a_j \right) - \frac{l \cdot k}{n} \right)^2 \cdot \left(\frac{1}{n(n-1)} \right) = \min_{b \in C_{k,n}} \{f(b)\}.$$

□

<u>Conjecture 3.1.11 (More Bjorklund Metric Properties):</u>

(a). $a \in C_{k,n}$ is minimally even $\Rightarrow \max\limits_{b \in C_{k,n}} \{f(b)\} = f(a)$,

(b). $f(a) = f(a^{-1})$.

4.0. General Evenness

General Evenness and Bjorklund's Metric:

Bjorklund's Evenness Metric:

While most of our discussion so far has pertained to the notion of maximal evenness, the notion of general rhythm evenness will fill this section and will arise in the Appendix in which an application of an arbitrary evenness measure to musical harmonic theory will be presented.

There are multiple metrics that can be used to calculate evenness on integer necklaces, or rhythms, (see [5] for a list) however only one called Bjorklund's evenness metric will be discussed here, originally presented in [9]. Bjorklund's metric considers rhythms in their binary representation and calculates the mean of the variances of all forward distances from and to each of the 1s. An equivalent variation of the Bjorklund metric defined in terms of the integer form of rhythms will be presented here. While a more efficient version of the Bjorklund metric calculable in linear time exists (see Appendix 4 in [10]), I will present a less efficient but more intuitive representation akin to that presented in [9]. The metric is as follows:

Definition 4.0.1 (Bjorklund's Metric):

$$f(a) = \frac{\sum_{i=0}^{n-1}\sum_{l=1}^{n-1}\left(\left(\sum_{j=i}^{i+l-1} a_j\right) - \frac{l\cdot k}{n}\right)^2}{n^2 - n},$$ where *a* is a rhythm in integer necklace form with length *n* and weight *k*.

From each index i, the sum of a_i and the proceeding $l-1$ elements is calculated, and since these values will vary depending on i and l, the mean sum of $\frac{lk}{n}$ is subtracted and this difference is squared. For each i, the variance of all l termed sums is calculated and then the mean of these variances is calculated to produce the evenness value of the rhythm. For example: Consider *a* = (2, 3, 5). Adding up the three integers, we determine that *k* = 10, and *n* = 3. Then

$$f_{10,3}((2,3,5)) = \frac{\sum_{i=0}^{2}\sum_{l=1}^{2}\left(\left(\sum_{l=i}^{i+l-1}(2,3,5)_l\right) - \frac{10\cdot l}{3}\right)^2}{6}$$

$$= \left(\frac{\left(2-\frac{10}{3}\right)^2 + \left(2+3-\frac{10\cdot 2}{3}\right)^2}{6}\right) + \left(\frac{\left(3-\frac{10}{3}\right)^2 + \left(3+5-\frac{10\cdot 2}{3}\right)^2}{6}\right)$$
$$+ \left(\frac{\left(5-\frac{10}{3}\right)^2 + \left(5+2-\frac{10\cdot 2}{3}\right)^2}{6}\right) = \frac{41}{54} + \frac{17}{54} + \frac{13}{27} = \frac{14}{9}$$

So, the evenness value of $a = (2, 3, 5)$ is $\frac{14}{7}$.

The primary benefit of this metric is that it distinguishes rhythms up to rotation and does not distinguish between rhythms in reverse order. It also assigns its minimum value to the Euclidean rhythm (See Proposition 3.1.10). A drawback of the Bjorklund metric is that it cannot distinguish between all differing rhythms with the same weight and length that are not reversals of one another: for example for $k = 12, n = 7$, (1,1,3,2,2,1,2), (1,1,3,1,3,1,2), and (1,1,3,1,1,3,2) each have an evenness value of $\frac{16}{21}$, but none are reversals of each other. Notice that (1,1,3,1,1,3,2) has more of a bimodal distribution of its integers than (1,1,3,2,2,1,2), so a question needs to be asked about what exactly evenness means here. Does a rhythm with a multimodal distribution of its elements count as more even than one with a larger unimodal distribution? The sum of their variances is the same, but the Bjorklund metric ignores how these variances are distributed. Fortunately, rhythms have the same modal distribution as their reversals, so a metric that distinguishes between rhythms with different modal distributions would still have the same benefits of the Bjorklund metric. To conclude, it appears that for any weight and length, the definition of maximally even rhythms, see Definition 1.1.3.1, is well established. However, a more precise definition of evenness to all rhythms needs to be established and a metric that reflects this precision should be found or constructed. To illustrate an application of such a metric to musical harmonic theory, a mathematical construction is presented in the following section.

5.0. An Application of Evenness to Musical Harmonic Theory

5.0.0. Introduction:

In Western music harmony, under the musical assumption of “enharmonic equivalence”, all notes can be classified into twelve classes called pitch classes (pcs). These pitch classes can be represented as integers from {0, 1, …, 11}, and there are two schemes that indicate which pcs should be represented by which integers. The first scheme is the chromatic scheme, where there is a semitone interval between consecutive integers such that 11 and 0 are consecutive because there are only twelve pcs. This scheme appears to be the preferred one in the literature known to the current author at this time. The second scheme seems less used, and it is called the circle of fifths scheme (COF). The circle of fifths scheme assigns pcs to the integers such that consecutive integers are seven semitones apart. Note that 7 is a generator of the additive group Z_{12}, so proceeding along consecutive integers from 0 twelve times puts us back at 0 and all pcs have a unique integer correspondent. Given this modular structure, we can imagine the twelve integers arranged on a circle in clockwise ascending order, hence the name "Circle of fifths".

This second scheme is to be used below because the measure of evenness on it appears to be more applicable to music harmony. A fact of music harmony says that adjacent pcs along the COFs are more consonant when played together or close in time to one another. Extending this fact, we can assume that any number of pcs, that are closer rather than farther to one another, along the COFs, are more consonant than dissonant. This is where the evenness measure comes in, because if we imagine the COFs as a length 12 binary rhythm, where 1s denote pcs played and 0s denote pcs not played, then we get the elements of $C_{12,n}$ representing n-note chords or scales in the most general form. For example, (1,3,8) in $C_{12,3}$ corresponds to the "Major triad" and its mirror image, (1,8,3), corresponds to the "Minor triad", (1,1,1,1,1,1,6) in $C_{12,7}$ corresponds to the "Diatonic scale", and (4,4,4) in $C_{12,3}$ corresponds to the "Augmented triad". Interestingly, the more even a chord/scale is, there is more relative separation between all pcs in the chord/scale along the COFs, and vice versa. This means that if a chord/scale is more uneven, then it is more consonant, and if it is more even, then it is more dissonant. Therefore, under the COFs scheme and the above assumption, the chord/scale represented by the maximally even composition (Euclidean rhythm) is the most dissonant chord/scale for a given *n*.

It appears therefore that evenness can be used with pcs as variables to construct a musical harmonic space that describes both the level of consonance of a chord as well as the pc it is based on. Basically, this is a pc by consonance space where consonance is measured by some evenness measure. Since the pc axis is modular, it can be represented as a polar axis, and since every $C_{k,n}$ is finite, evenness can be represented as a radial axis, with the center being maximally even. What results is a finite discrete disk space where points denote collections of subsets of {0, 1, …, 11}, representing chords/scales based at some pc that have a particular evenness value.

What is more, since $C_{k,n}$ are equivalent to integer rhythms, they can also represent musical rhythms. We can then analyze the evenness of musical rhythms by partitioning $C_{k,n}$ by some evenness measure. [7] showed that Euclidean rhythms (maximally even) are extraordinarily common in music, so perhaps this is because more even rhythms are easier to listen to and understand. It would be interesting to classify musical rhythms based on evenness to determine a more precise conception of the musical connection to evenness.

Below is a construction of such a space for any *k* and *n*. The hope is that once a satisfactory evenness measure is determined, the following space or something similar can be used by musicians as a theory to rationalize harmonic modulations in their music.

<u>Partitioning $C_{k,n}$ using Evenness measure:</u>

<u>1. (Defining the Evenness Measure):</u>

$f: C_{k,n} \to \mathbb{R}^+$ given by $f(a), a \in C_{k,n}$.

<u>2. (Defining Equivalence Relation under this Measure):</u>

$\forall\ a, b \in C_{k,n}, a \sim_f b \Leftrightarrow f(a) = f(b)$.

<u>3. (Partitioning Composition Set):</u>

Define $C'_{k,n} = \dfrac{C_{k,n}}{\sim_f}$.

<u>Musical Motivation:</u>

- Since we can partition $C_{k,n}$ into evenness classes, and $C_{k,n}$ represents musical rhythms and chords/scales, we can classify rhythms and chords/scales in terms of their evenness.
- The more even a rhythm is, the less complicated it is to listen to.
- The more even a chord/scale is, the more dissonant it sounds.
- These compositions are efficient representations of musical rhythms and chords/scales.

<u>Construction of a Disk Space that may have Applications to Musical Harmonic Theory:</u>

Part 1: Construction of the Evenness Radial Axis:

<u>1.1. (Defining Note Set Collection and Relating Ck,n to These Note Sets):</u>

Define the collection of note sets with n notes from k pitch classes as

$N = \left\{ s = \{0, 1,..., k-1\}^n : \forall\ i, j, s_i \neq s_j \right\}$.

$\forall\ s, t \in N$, let $N_s = \left(s_1 - s_n, s_2 - s_1, ..., s_n - s_{n-1}\right), N_t = \left(t_1 - t_n, t_2 - t_1, ..., t_n - t_{n-1}\right)$ be necklaces. Note that these necklaces are also elements of $C_{k,n}$.

Therefore, define the following relation:

$\forall\ s, t \in N, s \sim_E t \Leftrightarrow f\left(N_s\right) = f\left(N_t\right)$. Here, s and t are said to be equivalent under the evenness metric f .

<u>1.2. (Note Set Collection Partitioned into Evenness Classes):</u>

Define $N^E = \dfrac{N}{\sim_E}$. So N^E is the collection of all note sets partitioned into evenness classes determined by $\sim_E$. Therefore $\left(N^E, f\right)$ also defines a metric space.

Part 2: Construction of the "Tonal Mean" Polar Axis:

2.1. (Defining "Tonal Mean" Relation):

$\forall\, s, t \in N, s \sim_T t \Leftrightarrow \exists\, m \in \{0, 1, ..., k-1\}$ *s.t.*
$\min_{m_s \in \{0,...,k-1\}} \left\{ \sum_{i=1}^{n} \left| m - s_i \right| \right\} = \min_{m_t \in \{0,...,k-1\}} \left\{ \sum_{i=1}^{n} \left| m - t_i \right| \right\} = m$, where $m = m_s = m_t$. Here, s and t are said to have the same tonal mean of m.

Tonal mean is a way of picking a note between 0 and $k - 1$ so that all the notes of a note set are centered at the tonal mean.

2.2. (Partitioning N into Tonal Mean Classes):

Define $N^T = \dfrac{N}{\sim_T}$. Then N^T is the set of classes of note sets with equal tonal mean. Note that since there are k possible tonal means, N^T has size k .

Part 3: (Joining the Axes together):

3.1. (Definition of Product Set):

Define $H_{k,n} = \{(e, m) \in N^E \times N^T : (E, T)$ is the collection of note sets with evenness value e and tonal mean $m\}$.

This set constructs a finite discrete disk space, with radial axis denoting evenness described by f and polar axis denoting “tonal mean”. The purpose of defining this space is so that when given an evenness value and tonal mean, a point on the disk is described, and this point corresponds to all note sets (chords and scales) that have the given evenness value and tonal mean.

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